Related Rates Day 1 notes

Steps for Solving Related Rates Problems

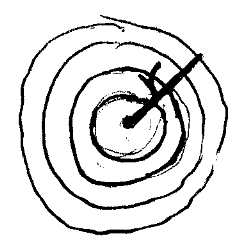
A. Read the problem carefully and identify all the quantities that are changing (either increasing or decreasing).
B. Use appropriate variables to represent the quantities involved in the problem. Write down all equations that relate these variables and other given constants. These equations might involve area or volume formulas, the Pythagorean theorem, or similar triangles from geometry, or other formulas from chemistry or physics.
C. Write a single equation that involves only the quantities whose rates of change are given and the one quantity whose rate of change is unknown.
D. Assume that all the variables in this equation are functions of time, usually given implicitly, and differentiate both sides of the equation with respect to time t; ie., $\frac{d}{dt}$. This usually involves implicit differentiation.
E. In the resulting related rate equation, substitute the given numerical quantities and rates. Remember that a rate of change of a quantity is a positive number if that quantity is increasing, and is a negative number if that quantity is decreasing.
F. Solve for the unknown rate of change. Be sure to include the appropriate units of measurement with your answer.

Example 1

A pebble is dropped into a calm pond, causing ripples in the form of concentric circles. The radius r of the outer ripple is increasing at a constant rate of 1 foot per second. When the radius is 4 feet, at what rate is the total area A of the disturbed water changing?

$\frac{dr}{dt} = \underline{1}\ \text{ft/sec}$

Find $\frac{dA}{dt}$ when $r = 4\text{ft}$.



$A = \pi r^2$

$\frac{dA}{dt} = \pi \cdot 2r \frac{dr}{dt}$

$\frac{dA}{dt} = \pi \cdot 2(\underset{ft}{4})(\underset{ft/sec}{1}) = 8\pi\ \frac{ft^2}{sec}$

the total area of the disturbed water is increasing at a rate of 8π ft^2/sec when $r = 4$ ft.

Example 2

Air is being pumped into a spherical balloon at a rate of 4.5 cubic inches per minute. Find the rate of change of the radius when the radius is 2 inches.

$\frac{dV}{dt} = 4.5 \text{ in}^3/\text{min}$

Find $\frac{dr}{dt}$ when $r = 2$in

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \frac{dr}{dt}$

$4.5 = \frac{4}{3}\pi \cdot 3(2)^2 \frac{dr}{dt}$

$\frac{9}{2} = 16\pi \frac{dr}{dt}$

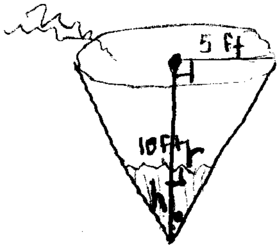
$\frac{dr}{dt} = \frac{\frac{9}{2}}{16\pi} = \frac{9}{32\pi} \text{ in}/\text{min}$

the radius is increasing at a rate of $\frac{9}{32\pi}$ in/min when $r = 2$in

Example 3

Water runs into a conical tank at the rate of 2 ft^3/min. The tank stands with the point down and has a height of 10 feet and a base diameter of 10 feet. How fast is the water level rising when the water is 6 feet deep?

$r = 5$ft



$\frac{dV}{dt} = 2 \text{ ft}^3/\text{min}$

Find $\frac{dh}{dt}$ when $h = 6$ft

$\frac{5}{10} = \frac{r}{h}$

$10r = 5h$

$r = \frac{5h}{10} = \frac{h}{2}$

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h = \frac{\pi}{12} h^3$

$\frac{dV}{dt} = \frac{\pi}{12} \cdot 3h^2 \frac{dh}{dt}$

$2 = \frac{\pi}{12} \cdot 3(6)^2 \frac{dh}{dt}$

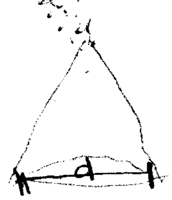
$2 = 9\pi \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{2}{9\pi} \text{ ft}/\text{min}$

the water level is rising at a rate of $\frac{2}{9\pi}$ ft/min when $h = 6$ ft

Example 4

Sand falls from a conveyor belt at the rate of 10 m^3/min onto the top of a conical pile. The height of the pile is always 3/8 of the base of the diameter. How fast are the height and radius changing when the pile is 4 m high?



$\frac{dV}{dt} = 10 \text{ m}^3/\text{min}$

$h = \frac{3}{8}d = \frac{3}{8}(2r) = \frac{6r}{8} = \frac{3r}{4}$

$4h = 3r$

$r = \frac{4h}{3}$

Find $\frac{dh}{dt}$ and $\frac{dr}{dt}$ when $h = 4$m

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \left(\frac{4h}{3}\right)^2 h = \frac{16\pi}{27} h^3$

$\frac{dr}{dt} = \frac{4}{3} \cdot \frac{dh}{dt}$

$\frac{dV}{dt} = \frac{16\pi}{27} \cdot 3h^2 \frac{dh}{dt}$

$10 = \frac{16\pi}{27} \cdot 3(4)^2 \frac{dh}{dt}$

$10 = \frac{256\pi}{9} \cdot \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{90}{256\pi} = \frac{45}{128\pi} \text{ m}/\text{min}$

$\frac{dr}{dt} = \frac{4}{3} \cdot \frac{45}{128\pi}$

$\frac{dr}{dt} = \frac{15}{32\pi} \text{ m}/\text{min}$

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the ht. is incr. at a rate of $\frac{45}{128\pi}$ m/min & the radius is incr. at a rate of $\frac{15}{32\pi}$ m/min when $h = 4$m

Example 5

The radius of a right circular cylinder is increasing at a rate of 2 *in/min* and the height is decreasing at a rate of 3 *in/min*. At what rate is the volume changing when the radius is 8 *in* and the height is 12 *in*? Is the volume increasing or decreasing?

$\frac{dr}{dt} = 2 \text{ in/min}$

$\frac{dh}{dt} = -3 \text{ in/min}$

Find $\frac{dV}{dt}$ when $r = 8 \text{ in}$ & $h = 12 \text{ in}$

$V = \pi r^2 h$ use product rule

$\frac{dV}{dt} = \pi \left[(r^2)\left(\frac{dh}{dt}\right) + (h)\left(2r \frac{dr}{dt}\right) \right]$

$\frac{dV}{dt} = \pi \left[8^2 \cdot -3 + 12 \cdot 2 \cdot 8 \cdot 2 \right]$

$\frac{dV}{dt} = 192\pi \ \text{in}^3/\text{min}$

Example 6

A conical tank with vertex down has a radius of 10 ft at the top and is 24 ft high. If water flows out of the tank at a rate of 20 ft^3/min, how fast is the depth of the water decreasing when the water is 16 ft deep?

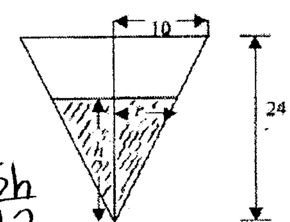
$\frac{dV}{dt} = -20 \text{ ft}^3/\text{min}$

Find $\frac{dh}{dt}$ when $h = 16 \text{ ft}$

$\frac{10}{24} = \frac{r}{h}$

$24r = 10h$

$r = \frac{10h}{24} = \frac{5h}{12}$



$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \left(\frac{5h}{12}\right)^2 h = \frac{25\pi}{432} h^3$

$\frac{dV}{dt} = \frac{25\pi}{432} \cdot 3h^2 \frac{dh}{dt}$

$-20 = \frac{25\pi}{432} \cdot 3(16)^2 \frac{dh}{dt}$

$-20 = \frac{400\pi}{9} \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{-9}{20\pi} \text{ ft/min}$

the depth of the water is decreasing at a rate of $\frac{9}{20\pi}$ ft/min when $h = 16$ ft

Example 7

Gas is escaping from a spherical balloon at the rate of 2 ft^3/min. How fast is the surface area changing when the radius is 12 *ft*?

$\frac{dV}{dt} = -2 \text{ ft}^3/\text{min}$

Find $\frac{dS}{dt}$ when $r = 12 \text{ ft}$

$S = 4\pi r^2$

$\frac{dS}{dt} = 4\pi \cdot 2r \frac{dr}{dt}$

$\frac{dS}{dt} = 4\pi \cdot 2(12)\left(-\frac{1}{288\pi}\right) = -\frac{1}{3} \text{ ft}^2/\text{min}$

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \frac{dr}{dt}$

$-2 = \frac{4}{3}\pi \cdot 3(12)^2 \cdot \frac{dr}{dt}$

$-2 = 576\pi \cdot \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{-2}{576\pi} = -\frac{1}{288\pi}$

the S.A. is decr at a rate of $\frac{1}{3}$ ft^2/min when $r = 12$ ft