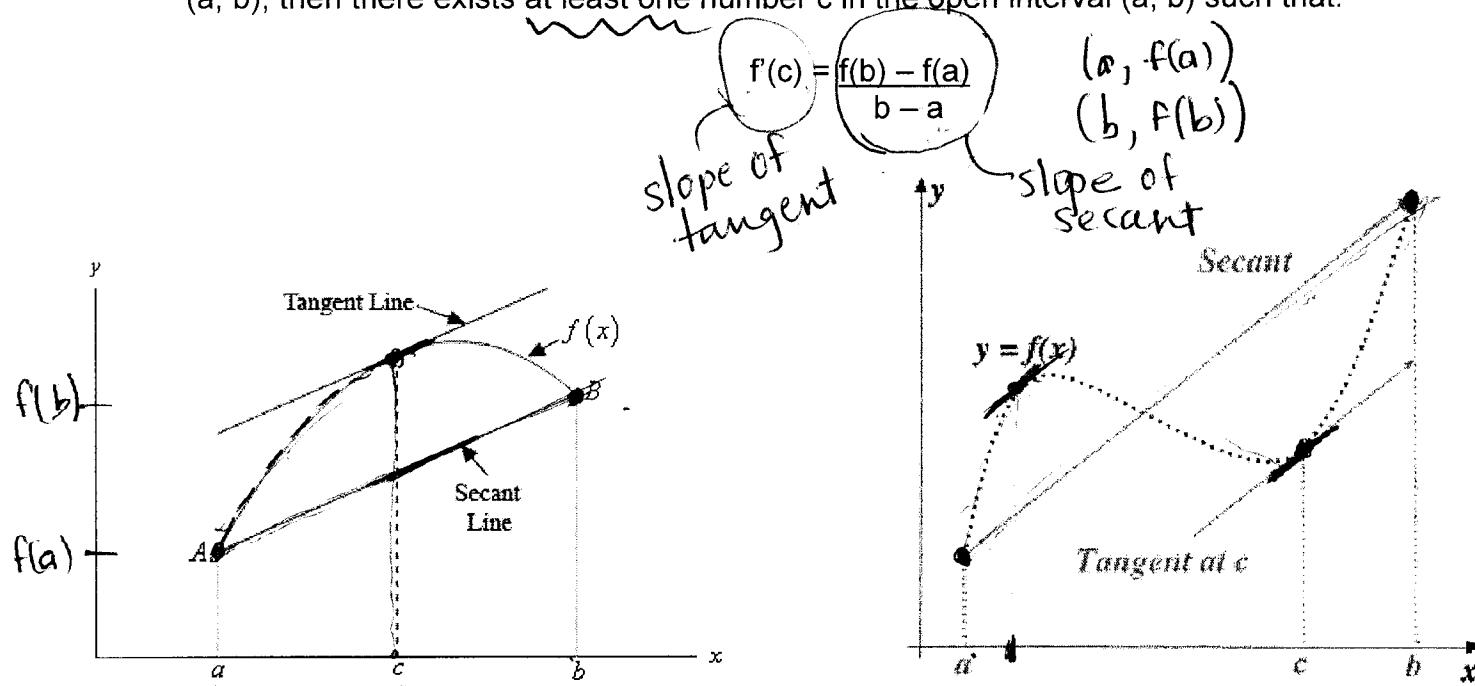


AP Calculus AB Unit #3 Notes

Applications of Derivatives: Important Theorems

Mean Value Theorem

If the function f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) , then there exists at least one number c in the open interval (a, b) such that:



Example #1:

If the function f is defined on $[1, 3]$ by $f(x) = 4 - 3/x$, show that the MVT can be applied to f and find a number c which satisfies the conclusion:

$$\textcircled{1} \quad f \text{ is cont on } [1, 3] \quad \checkmark$$

$$f'(x) = 0 - 3 \cdot -1x^{-2} = \frac{3}{x^2}$$

$$\textcircled{2} \quad f \text{ is diff on } (1, 3) \quad \checkmark$$

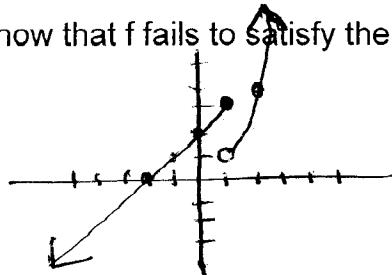
$$\text{points } (1, 1), \quad \frac{3}{x^2} = \frac{3-1}{3-1} = 1 \quad x = \pm \sqrt{3}$$

Example #2:

Sketch a graph of the function f if $f(x) = \begin{cases} x+2, & \text{for } x \leq 1 \\ x^2, & \text{for } x > 1 \end{cases}$

$$C = \sqrt{3}$$

Show that f fails to satisfy the MVT on the interval $[-2, 2]$.



f is not cont on $[-2, 2] \Rightarrow$ MVT doesn't apply
(jump when $x=1$)

Example #3:

Suppose that $s(t) = t^2 - t + 4$ is the position of the motion of a particle moving along a line.

a) Explain why the function s satisfies the hypothesis of the MVT.

① $s(t)$ is cont. on any interval

② $s(t)$ is diff. on any interval

b) Find the value of t in $[0, 3]$ where instantaneous velocity is equal to the average velocity.

$$s'(t) = 2t - 1 \quad \text{points } (0, 4), (3, 10)$$

$$2t - 1 = \frac{10 - 4}{3 - 0} = 2$$

$$\frac{2t - 1}{t} = \frac{3}{2}$$

Rolle's Theorem

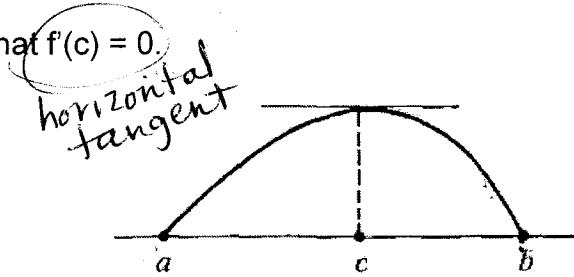
Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$
2. f is differentiable on the open interval (a, b)
3. $f(a) = f(b)$

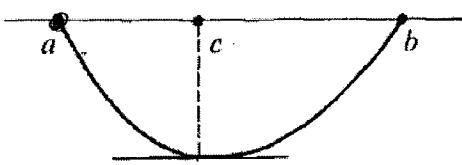
Then there is a number c in (a, b) such that $f'(c) = 0$.



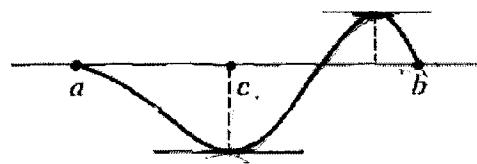
Case 1, $m = M = 0$



Case 2, $M > 0$



Case 3, $m < 0$



Case 2 and 3, $m < 0 < M$

Example #4:

Verify the function satisfies the three hypotheses of Rolle's Theorem on the given interval. Then find all numbers c that satisfy the conclusion of Rolle's Theorem.

a) $f(x) = x^2 - 4x + 1, [0, 4]$

- ① f is cont on $[0, 4]$ ✓
- ② f is diff on $(0, 4)$ ✓
- ③ $f(0) = f(4)$ ✓

$$f'(x) = 2x - 4 = 0$$

$$x = 2$$

$$\boxed{c = 2}$$

b) $f(x) = \sin(2\pi x), [-1, 1]$

- ① f is cont on $[-1, 1]$ ✓
- ② f is diff on $(-1, 1)$ ✓
- ③ $f(-1) = f(1)$ ✓

$$f'(x) = \cos(2\pi x) \cdot 2\pi = 0$$

$$\cos(2\pi x) = 0$$

$$2\pi x = \cos^{-1}(0)$$

$$2\pi x = \frac{\pi}{2}, \frac{3\pi}{2}, -\frac{\pi}{2}, -\frac{3\pi}{2}$$

Example #5:

Let $f(x) = 1 - x^{2/3}$. Show that $f(-1) = f(1)$ but there is no number c in $(-1, 1)$ such that $f'(c) = 0$.

Why does this contradict Rolle's Theorem?

$$f(-1) = 1 - (-1)^{2/3} = 1 - 1 = 0$$

$$f(1) = 1 - (1)^{2/3} = 0$$

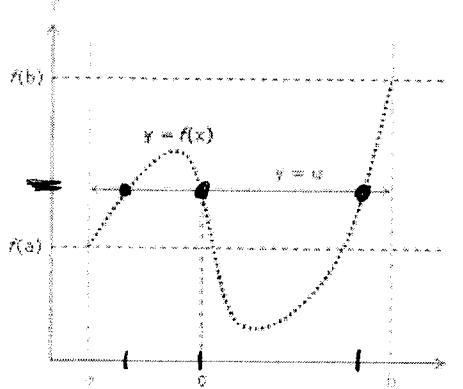
$$f'(x) = 0 - \frac{2}{3}x^{-\frac{1}{3}} = \frac{-2}{3\sqrt[3]{x}}$$

$f'(x)$ is not defined at $x=0 \Rightarrow$

f is not diff on $(-1, 1)$

Intermediate Value Theorem

If f is a continuous function on the closed interval $[a, b]$, with $f(a) \neq f(b)$, and k is a number between $f(a)$ and $f(b)$, then there exists at least one number c in (a, b) for which $f(c) = k$.



y-value
x-value

Example #6

Use the Intermediate Value Theorem to show that there is a zero for the given function in the specified interval.

a) $f(x) = x^3 - 3x + 1$, $[0, 1]$

- ① f is cont on $[0, 1]$ ✓
- ② $f(0) \neq f(1)$ $1 \neq -1$ ✓
- ③ 0 is between $f(0) \in f(1)$
 $1 \quad -1$ ✓

y is zero

there is an x-value, "c",
so that $f(c) = 0$

b) $g(x) = \ln(x) - e^{-x}$, $[1, 2]$

- ① g is cont. on $[1, 2]$ ✓
- ② $g(1) \neq g(2)$ ✓ $\begin{aligned} g(1) &= \ln 1 - e^{-1} = -\frac{1}{e} \\ g(2) &= \ln 2 - e^{-2} \end{aligned}$
- ③ 0 is between $g(1) \in g(2)$
 $-0.368 \quad 0.558$ ✓

there is an
x-value "c"
so that $f(c) = 0$