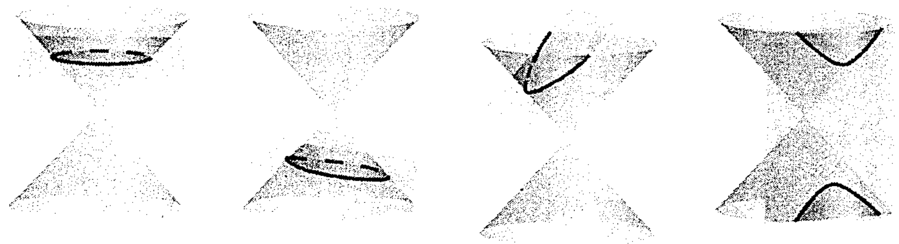
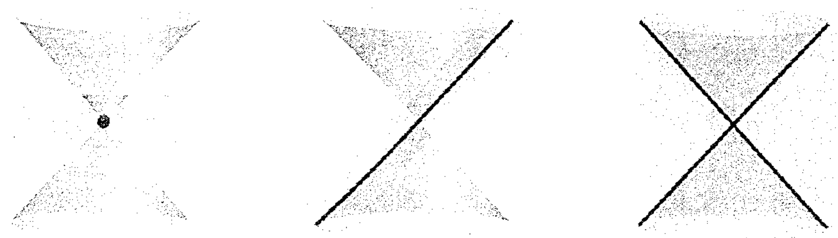
§9.1: Introduction to Conics: Parabolas

Conics

- A **conic section** (or simply **conic**) is the intersection of a plane and a double-napped cone
- In the formation of the four **basic conics**, the intersecting plane does not pass through the vertex of the cone
- When the plane does pass through the vertex, the resulting figure is a **degenerate conic**



Circle Ellipse Parabola Hyperbola

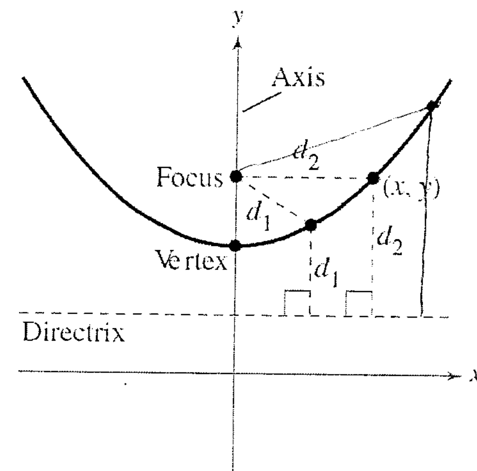


Point line Two intersecting lines

Definition of a Parabola

A **parabola** is the set of all points (x, y) in a plane that are equidistant from a fixed line, **the directrix,** and **a** fixed point, the **focus,** not on the line. (See Figure 9.3.) The midpoint between the focus and the directrix is the **vertex,** and the line passing through the focus and the vertex is the **axis** of the parabola.

- The **axis (of symmetry)** affects the symmetry of the parabola.
- The **focus** affects the size of the parabola.
- The **directrix** affects the direction of the parabola.
- The **vertex** affects the location of the parabola.



The General Form of a Parabola

- The general second-degree equation for *all* conics is $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$.
- For parabolas, $B = 0$, and <u>either</u> $A = 0$ or $C = 0$ (but not both).
  - If $C = 0$: $Ax^2 + Dx + Ey + F = 0$ (parallel to *x*-axis, opens up/down)
  - If $A = 0$: $Cy^2 + Dx + Ey + F = 0$ (parallel to *y*-axis, opens left/right)

$$\frac{1}{4p}(x-h)^2 = y-k$$

$$y = \frac{1}{4p}(x-h)^2 + k$$

$$y = a(x-h)^2 + k \qquad a = \frac{1}{4p}$$

Standard Equation of a Parabola

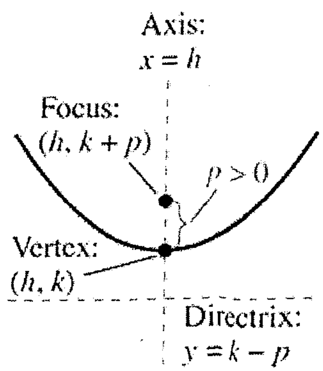
The **standard form of the equation of a parabola** with vertex at (h, k) is as follows.

$(x-h)^2 = 4p(y-k),\ p \neq 0$ — Vertical axis; directrix: $y = k - p$

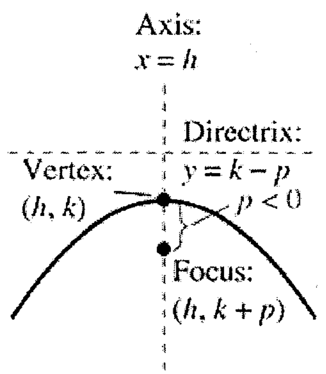
$(y-k)^2 = 4p(x-h),\ p \neq 0$ — Horizontal axis; directrix: $x = h - p$

4p is recognized as the Focal Chord Length.
This is a chord parallel to the directrix through the focus.

Some equations alternatively use c instead of p to represent distance to the focus.

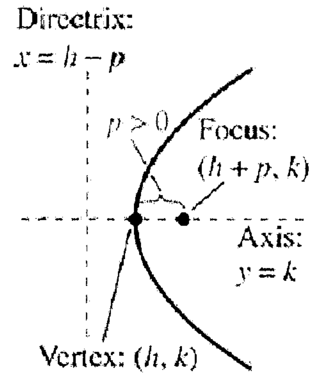


$(x-h)^2 = 4p(y-k)$
(a) Vertical axis: $p > 0$

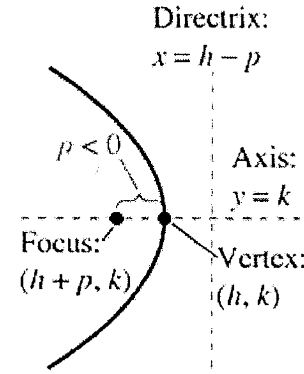


$(x-h)^2 = 4p(y-k)$
(b) Vertical axis: $p < 0$

- Note that x is squared
- The axis of symmetry is vertical
- The directrix is horizontal



$(y-k)^2 = 4p(x-h)$
(c) Horizontal axis: $p > 0$



$(y-k)^2 = 4p(x-h)$
(d) Horizontal axis: $p < 0$

- Note that y is squared
- The axis of symmetry is horizontal
- The directrix is vertical

Example 1

Convert $x^2 - 12x - 2y + 20 = 0$ into standard form.

$$x^2 - 12x = 2y - 20$$

$$\left(\frac{-12}{2}\right)^2 = 36 \qquad (x-6)(x-6) = 2y + 16$$

$$(x-6)^2 = 2(y+8)$$

$$y = \frac{1}{2}(x-6)^2 - 8 \qquad \text{Vertex } (6, -8)$$

$$4p = 2$$

$$p = \frac{2}{4} = \frac{1}{2}$$

Graphing a Parabola

- Convert the equation into standard form.
- Plot the vertex (h, k).
- Determine the value of p.
- Determine the direction of the directrix (which variable is squared?).
  - If the directrix is horizontal, it opens up ($p > 0$) or down ($p < 0$)
  - If the directrix is vertical, it opens right ($p > 0$) or left ($p < 0$)
- Plot the directrix and focus.
- Draw the parabola (opening away from the directrix).

Example 2

Graph $x^2 - 12x - 2y + 20 = 0$.

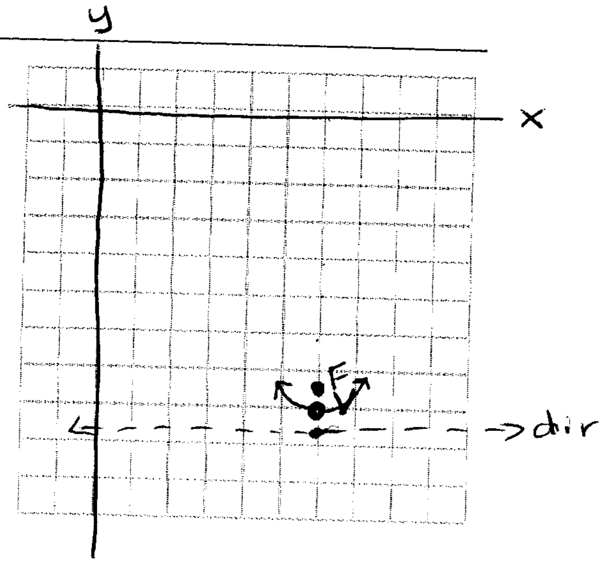
See ex #1

$V(6,-8)$

$(x-6)^2 = 2(y+8)$

$p = \frac{1}{2}$

opens up



focus $(6, -7.5)$

directrix $y = -8.5$

Example 3

Graph $4y + 16x = 44 - y^2$.

$y^2 + 4y = -16x + 44$

$\left(\frac{4}{2}\right)^2 = 4$

$(y+2)^2 = -16x + 48$

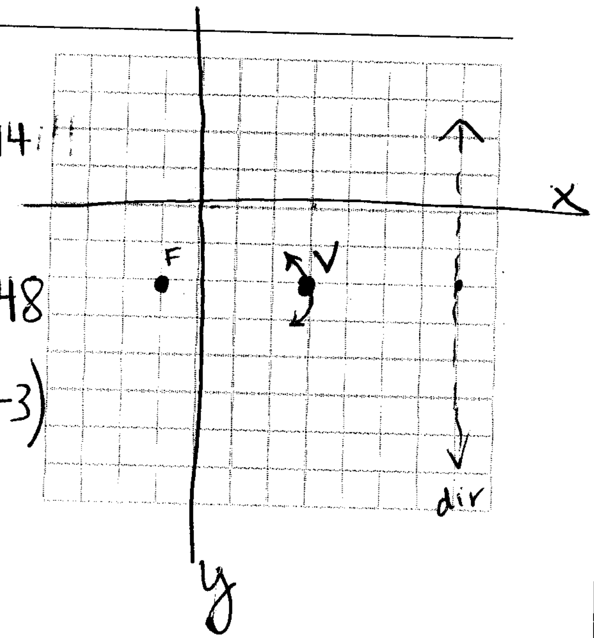
$(y+2)^2 = -16(x-3)$

$V(3,-2)$

$4p = -16$

$p = -4$

opens left



focus $(-1,-2)$

dir. $x = 7$

Example 4

Graph $x^2 - 6x + 8y + 41 = 0$.

$x^2 - 6x \quad = -8y - 41$

$\left(\frac{-6}{2}\right)^2 = 9$

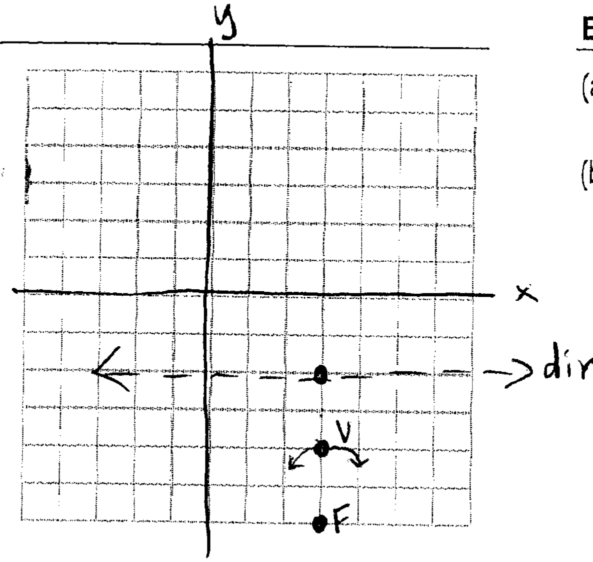
$(x-3)^2 = -8y - 32$

$(x-3)^2 = -8(y+4)$

$V(3,-4)$

$4p = -8$

$p = -2$



focus $(3,-6)$

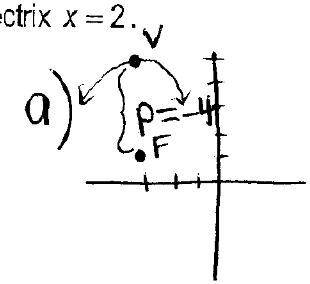
dir $y = -2$

Example 5

(a) Write the equation in standard form of a parabola with vertex $(-3,5)$ and focus at $(-3,1)$.

(b) Write the equation in standard form of a parabola with vertex $(4,-1)$ and directrix $x = 2$.

a)



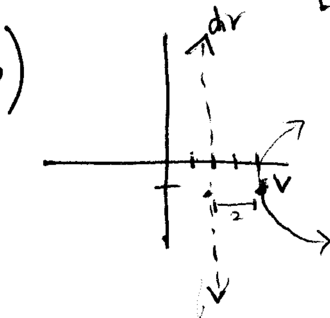
$V(\overset{h}{-3},\overset{k}{5})$

$p = -4$

$(x-h)^2 = 4p(y-k)$

$\boxed{(x+3)^2 = -16(y-5)}$

b)



$V(\overset{h}{4},\overset{k}{-1})$

$p = 2$

$(y-k)^2 = 4p(x-h)$

$\boxed{(y+1)^2 = 8(x-4)}$

> Definition of a Circle
>
> A **circle** is the set of all points (x, y) in a plane that are equidistant from a fixed point (h, k), called the **center** of the circle. (See Figure 9.3.) The distance r between the center and any point (x, y) on the circle is the **radius.**

The General Form of a Circle

- Circles are special forms of ellipses, where $A = C$
    - $Ax^2 + Cy^2 + Dx + Ey + F = 0$

> Standard Form of the Equation of a Circle
>
> The **standard form of the equation of a circle** is
>
> $(x - h)^2 + (y - k)^2 = r^2.$
>
> The point (h, k) is the center of the circle, and the positive number r is the radius of the circle. The standard form of the equation of a circle whose center is the origin, $(h, k) = (0, 0)$, is
>
> $x^2 + y^2 = r^2.$

- Similar to the equation of an ellipse: $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$
- For circles, $a^2 = b^2 = r^2$
- Since $a = b$, $c = 0$ and the eccentricity of a circle is $e = 0$

Graphing a Circle

- Convert the equation into standard form.
- Plot the center (h, k).
- Plot the points $(h \pm r, k)$ and $(h, k \pm r)$.
    - Always go r units in both directions on both axes.
- Connect the outer points as a smooth circle.

Example 5

Graph $x^2 + y^2 - 6x + 8y + 21 = 0$.

$x^2 - 6x + y^2 + 8y = -21$

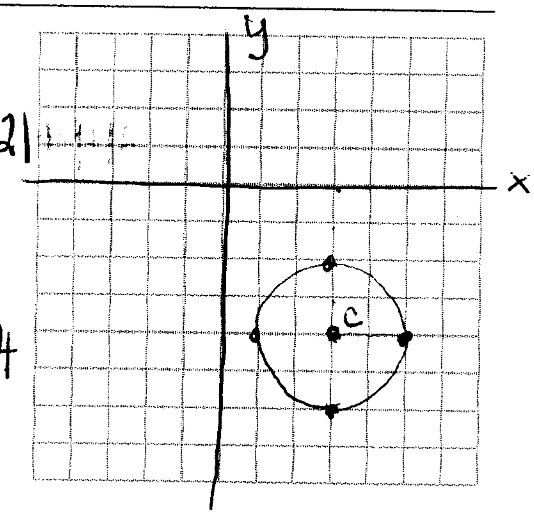
$\left(\frac{-6}{2}\right)^2 = 9 \quad \left(\frac{8}{2}\right)^2 = 16$

$(x-3)^2 + (y+4)^2 = 4$

Center $(3, -4)$

$r^2 = 4$

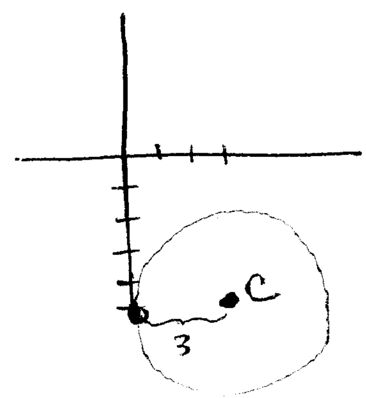
$r = 2$



Example 6

Write the equation in standard form of a circle with center $(3, -5)$ that is tangent to the y-axis.

h k



$r = 3$

$\boxed{(x-3)^2 + (y+5)^2 = 9}$