# Derivatives of Inverse Functions - Classwork

General Problem: Find the derivative of the inverse function of $f(x)$ at $x = k$.

**Method 1**: Simply finding the inverse function. This works when it is easy to generate the inverse function.

a) Find the inverse function by interchanging $x$ and $y$ and solving for $y$
b) Take the derivative of this new $y$. That will be the derivative of the inverse function.
c) Plug in your given $k$ value

**Method 2:** Not finding the inverse function because it is too difficult

a) Find the inverse function by interchanging $x$ and $y$ .

b) find $\frac{dy}{dx}$ implicitly

c) Solve for $\frac{dy}{dx}$. It will be in terms of $y$.

d) Replace the value of $k$ for $x$ in your inverse function from step a above and solve for $y$

e) Plug that value of $y$ into $\frac{dy}{dx}$

**Example:** If $f(x) = x^2, x \geq 0$, find the derivative of $f^{-1}(x)$ at $x = 4$

**METHOD 1**

a) $y = x^2$, so the inverse is $x = y^2$

therefore $y = \sqrt{x}$ (first quadrant)

b) $y' = \frac{1}{2\sqrt{x}}$

c) $y'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

**METHOD 2**

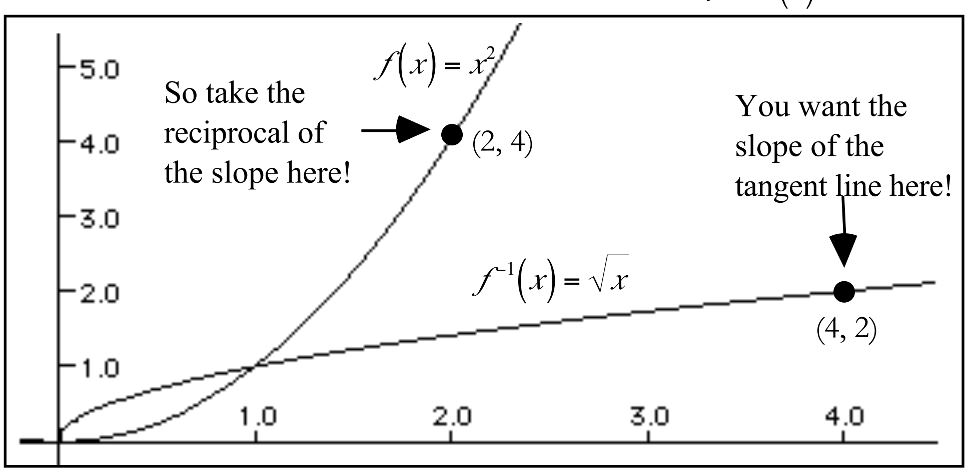
a) $y = x^2$, so the inverse is $x = y^2$

b) $1 = 2y\frac{dy}{dx}$

c) $\frac{dy}{dx} = \frac{1}{2y}$

d) $4 = y^2 \Rightarrow y = 2(\text{quad I})$

e) $\frac{dy}{dx} = \frac{1}{2y} = \frac{1}{2(2)} = \frac{1}{4}$



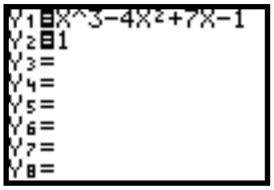

**Note:** It was necessary to restrict the domain of $f(x)$ to $x \geq 0$ so that its inverse is a function: i.e. that $f(x)$ is one-to-one (passes the horizontal line test).

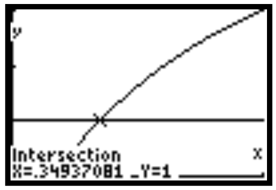
**Example**: Find the derivative of the inverse function of $f(x)=x^3-4x^2+7x-1$ at $x=1$.

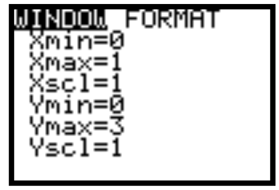
Method 1 will be too difficult. $y=x^3-4x^2+7x-1$ so the inverse is $x=y^3-4y^2+7y-1$

a) $1=\left(3y^2-8y+7\right)\dfrac{dy}{dx}\Rightarrow\dfrac{dy}{dx}=\dfrac{1}{3y^2-8y+7}$

b) Set $y^3-4y^2+7y-1=1$. Graphically, you get $y$= .349.



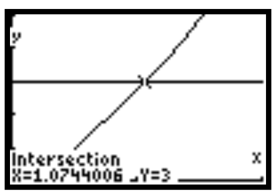

c) $\dfrac{dy}{dx}=\dfrac{1}{3(.349)^2-8(.349)+7}=.219$

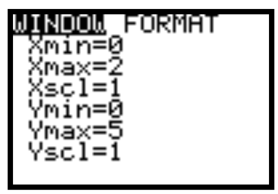
**Example**: Find the derivative of the inverse function of $f(x)=e^x+\ln x$ at $x$ = 3

Method 1 will be too difficult. $y=e^x+\ln x$ so the inverse is $x=e^y+\ln y$

a) $1=\left(e^y+\dfrac{1}{y}\right)\dfrac{dy}{dx}\Rightarrow\dfrac{dy}{dx}=\dfrac{1}{e^y+\dfrac{1}{y}}$ or $\dfrac{dy}{dx}=\dfrac{y}{ye^y+1}$

b) Set $e^y+\ln y=3$. Graphically, you get $y$ = 1.0744.


c) $\dfrac{dy}{dx}=\dfrac{1}{e^{1.0744}+\dfrac{1}{1.0744}}=.259$

*Note:* After you graphically intersect, you can easily get the answer by $\dfrac{1}{\text{nDeriv(Y1,X,X)}}$

**Example:** Find the derivative of the inverse function of $y=e^{x^2}$, $x$> 0

Inverse Function : $x=e^{y^2}$

Solution: $\ln x=y^2$

$y=\sqrt{\ln x}=(\ln x)^{1/2}$ $\quad\dfrac{dy}{dx}=\dfrac{1}{2}(\ln x)^{-1/2}\dfrac{1}{x}=\dfrac{1}{2x(\ln x)^{1/2}}$

**Sample Problems:** Find the derivative of the inverse function of (use method 2 only if method 1 won't work)

a) $y=x^3+1$ at $x=9$ $\quad\dfrac{dy}{dx}_{(x=2)}=\dfrac{1}{3x^2}=\dfrac{1}{12}$

b) $y=x^3+5x-1$ at $x=5$ $\quad\dfrac{dy}{dx}_{(x=1)}=\dfrac{1}{3x^2+5}=\dfrac{1}{8}$

c) $y=x+\sin x$ at $x=\pi$ $\quad\dfrac{dy}{dx}_{(x=\pi)}=\dfrac{1}{1+\cos x}=DNE$

# Derivatives of Inverse Functions - Homework

For the problems below, find the derivative of $f^{-1}$ for the function *f* at the specified value of *x*. No calculators.

1. $f(x) = x^3 + 2x - 1$ at $x = 2$

$$x^3 + 2x - 1 = 2 \Rightarrow x = 1$$
$$\frac{dy}{dx}_{(x=1)} = \frac{1}{3x^2 + 2} = \frac{1}{5}$$

2. $f(x) = 2x^5 + x^3 + 1$ at $x = 4$

$$2x^5 + x^3 + 1 = 4 \Rightarrow x = 1$$
$$\frac{dy}{dx}_{(x=1)} = \frac{1}{10x^4 + 3x^2} = \frac{1}{13}$$

3. $f(x) = \sin x \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$ at $x = \frac{1}{2}$

$$\sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}$$
$$\frac{dy}{dx}_{\left(x=\frac{\pi}{6}\right)} = \frac{1}{\cos x} = \frac{2}{\sqrt{3}}$$

4. $f(x) = \cos 2x \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$ at $x = 1$

$$\cos 2x = 1 \Rightarrow x = 0$$
$$\frac{dy}{dx}_{(x=0)} = \frac{-1}{2\sin 2x} = DNE$$

5. $f(x) = x^3 - \frac{4}{x} \quad x > 0$ at $x = 6$

$$x^3 - \frac{4}{x} = 6 \Rightarrow x = 2$$
$$\frac{dy}{dx}_{(x=2)} = \frac{1}{3x^2 + \frac{4}{x^2}} = \frac{1}{13}$$

6. $f(x) = \sqrt{x - 4}$ at $x = 2$

$$\sqrt{x-4} = 2 \Rightarrow x = 8$$
$$\frac{dy}{dx}_{(x=8)} = \frac{1}{\frac{1}{2\sqrt{x-4}}} = 4$$

For the problems below, find the derivative of $f^{-1}$ for the function *f* at the specified value of *x*. Use calculators.

7. $f(x) = x^3 - 2x^2 + 5x - 1$ at $x = 2$

$$\frac{dy}{dx}_{(x=.737)} = \frac{1}{3x^2 - 4x + 5} = .272$$

8. $f(x) = \sqrt[3]{3x - 5}$ at $x = -3$

$$\frac{dy}{dx}_{(x=-7.333)} = \frac{1}{\frac{1}{(3x-5)^{2/3}}} = 9$$

9. $f(x) = \frac{x}{2} + \sin^2 x$ at $x = 3$

$$\frac{dy}{dx}_{(x=4.309)} = \frac{1}{\frac{1}{2} + 2\sin x \cos x} = .818$$

10. $f(x) = xe^{\cos x}$ at $x = 3$

$$\frac{dy}{dx}_{(x=4.335)} = \frac{1}{xe^{\cos x}(-\sin x) + e^{\cos x}} = 0.287$$