

AP Calculus BC

Notes 2.1 Sequences

Sequence - a function whose domain is the set of positive integers. Its elements are called "terms". n is the number term and used to generate the terms. a_n is the general term or the "nth term".

n	1	2	3	4	5	$\frac{n}{n}$
a_n	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	

Arithmetic Sequence - has terms with a common difference

$$\text{Ex: } -3, 1, 5, 9, 13, \dots \quad d = 4 \quad \text{Ex: } 13, 7.5, 2, -3.5, \dots \quad d = -5.5$$

Geometric Sequence - has terms with a common ratio

$$\text{Ex: } 16, 8, 4, 2, 1, \dots \quad r = \frac{1}{2} \quad \text{Ex: } 2, 5, \frac{25}{2}, \frac{125}{4}, \dots \quad r = \frac{5}{2}$$

Limit of a Sequence

Let L be a real number. L is the limit of a sequence $\{a_n\}$, written $\lim_{n \rightarrow \infty} a_n = L$.

- If the limit EXISTS, then the sequence converges
- If the limit DNE, then the sequence diverges

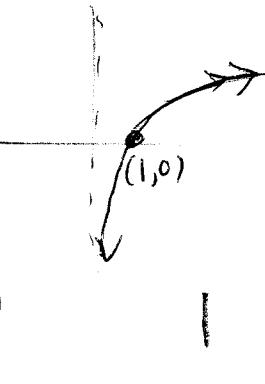
Ex 1) Given the n^{th} term, find the limit, if it exists.

a) $a_n = \frac{1}{n}$ $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ seq. converges

b) $a_n = \left(1 + \frac{1}{n}\right)^n$ $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \xrightarrow[n \rightarrow \infty]{\text{graph}} e$ seq. converges

$\lim_{n \rightarrow \infty} \ln \left(1 + \frac{1}{n}\right)^n = \lim_{n \rightarrow \infty} n \cdot \ln \left(1 + \frac{1}{n}\right)$ $\infty \cdot 0$

$\lim_{n \rightarrow \infty} \frac{\ln \left(1 + \frac{1}{n}\right)}{\frac{1}{n}} \stackrel{0}{0}$ $\lim_{n \rightarrow \infty} \frac{\frac{1}{1+n} \cdot -\frac{1}{n^2}}{-\frac{1}{n^2}} = \frac{1}{1+0} = 1$



c) $a_n = 3 + (-1)^n \quad 2, 4, 2, 4, 2, 4, \dots$

$$\lim_{n \rightarrow \infty} (3 + (-1)^n) = \boxed{\text{DNE}} \quad \text{seq. diverges}$$

odd terms $\rightarrow 2$
 even terms $\rightarrow 4$ \rightarrow different

d) $a_n = \sin n \quad \lim_{n \rightarrow \infty} \sin n \quad \leftarrow \begin{matrix} \text{sine} \\ \text{oscillates} \end{matrix}$

$$= \boxed{\text{DNE}} \quad \text{seq. diverges}$$

Ex 2) Determine convergence.

a) $a_n = \frac{1-5n^4}{n^4+8n^3}$

$$\lim_{n \rightarrow \infty} \frac{1-5n^4}{n^4+8n^3} = -5 \quad \boxed{\text{Seq. converges}}$$

b) $a_n = \frac{n^2-2n+1}{n-1}$

$$\lim_{n \rightarrow \infty} \frac{n^2-2n+1}{n-1} = \infty \quad \boxed{\text{Seq. diverges}}$$

c) $a_n = \left(\frac{n+1}{2n}\right)\left(1 - \frac{1}{n}\right)$

$$\lim_{n \rightarrow \infty} \left(\frac{n+1}{2n}\right)\left(1 - \frac{1}{n}\right) = \frac{1}{2} \cdot 1 = \frac{1}{2} \quad \boxed{\text{Seq. converges}}$$

d) $a_n = \frac{n^2}{2^n - 1}$

$$\lim_{n \rightarrow \infty} \frac{n^2}{2^n - 1} = 0 \quad \boxed{\text{Seq. converges}}$$

e) $a_n = 1 + \frac{(-1)^n}{n}$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{(-1)^n}{n}\right) = 1 \quad \boxed{\text{Seq. converges}}$$

odd terms: $0, \frac{2}{3}, \frac{4}{5}, \frac{6}{7}, \dots \rightarrow 1$

even terms: $\frac{3}{2}, \frac{5}{4}, \frac{7}{6}, \frac{9}{8}, \dots \rightarrow 1$ same

Properties of Limits of Sequences

Let $\lim_{n \rightarrow \infty} a_n = L$ and $\lim_{n \rightarrow \infty} b_n = K$

- $\lim_{n \rightarrow \infty} (a_n \pm b_n) = L \pm K$
- $\lim_{n \rightarrow \infty} ca_n = C \cdot \lim_{n \rightarrow \infty} a_n = C \cdot L$
- $\lim_{n \rightarrow \infty} (a_n b_n) = L \cdot K$
- $\lim_{n \rightarrow \infty} \left(\frac{a_n}{b_n}\right) = \frac{L}{K}, K \neq 0$

Squeeze Theorem for Sequences

If $\lim_{n \rightarrow \infty} a_n = L$ & $\lim_{n \rightarrow \infty} b_n = L$ and there exists an integer N such that $a_n \leq c_n \leq b_n$ for all $n > N$, then $\lim_{n \rightarrow \infty} c_n = L$.

Ex 3) Given $c_n = \frac{1}{2^n}$, find $\lim_{n \rightarrow \infty} c_n$ $\lim_{n \rightarrow \infty} \frac{1}{2^n} = 0$

$$\lim_{n \rightarrow \infty} 0 \leq \lim_{n \rightarrow \infty} \frac{1}{2^n} \leq \lim_{n \rightarrow \infty} \frac{1}{n}$$

↗

$$0 \leq \lim_{n \rightarrow \infty} \frac{1}{2^n} \leq 0$$

Absolute Value Theorem for Sequences

Given the sequence $\{a_n\}$. If $\lim_{n \rightarrow \infty} |a_n| = 0$, $\lim_{n \rightarrow \infty} a_n = 0$.

Ex 4) $a_n = -\frac{1}{n}$

$$\lim_{n \rightarrow \infty} \frac{-1}{n} = 0$$

$$\lim_{n \rightarrow \infty} \left| -\frac{1}{n} \right| = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

Ex 5) Given each sequence, find the n^{th} term.

n	1	2	3	4	5
a_n	2	$\frac{4}{3}$	$\frac{8}{5}$	$\frac{16}{7}$	$\frac{32}{9}$

n	1	2	3	4	5
a_n	-2	$\frac{8}{2}$	$-\frac{26}{6}$	$\frac{80}{24}$	$-\frac{242}{120}$

$$a_n = \frac{2^n}{2n-1}$$

$$a_n = \frac{(-1)^n (3^n - 1)}{n!}$$

$$n! = n(n-1)(n-2)(n-3)\dots(1)$$

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$$