

1972 AB5
Solution

(a) $y = 2e^{\cos x}$

$$\frac{dy}{dx} = 2e^{\cos x}(-\sin x) = -2(\sin x)e^{\cos x}$$

$$\frac{d^2y}{dx^2} = -2(\sin x)e^{\cos x}(-\sin x) - 2(\cos x)e^{\cos x} = 2e^{\cos x}(\sin^2 x - \cos x)$$

(b) $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = -2(\sin x)e^{\cos x} \cdot \frac{dx}{dt}$

Substituting $\frac{dy}{dt} = 5$ and $x = \frac{\pi}{2}$ gives

$$5 = -2 \left(\sin \frac{\pi}{2} \right) e^{\cos(\pi/2)} \frac{dx}{dt} = -2(1)e^0 \frac{dx}{dt} = -2 \frac{dx}{dt}.$$

Therefore $\frac{dx}{dt} = -\frac{5}{2}$ when $x = \frac{\pi}{2}$.

1993 AB2**Solution**

$$\begin{aligned}(a) \quad v(t) &= x'(t) = 2e^{-t} - 2te^{-t} \\ a(t) &= x''(t) = -2e^{-t} - 2e^{-t} + 2te^{-t} \\ a(0) &= x''(0) = -2 - 2 = -4\end{aligned}$$

$$\begin{aligned}(b) \quad x''(t) &= -2e^{-t}(2-t) = 0 \\ t &= 2 \\ x'(2) &= v(2) = 2e^{-2}(1-2) \\ &= \frac{-2}{e^2} \\ &\approx -0.271\end{aligned}$$

$$\begin{aligned}(c) \quad x'(t) &= v(t) = 2e^{-t}(1-t) = 0 \\ t &= 1 \\ x(0) &= 0 \\ x(1) &= \frac{2}{e} \approx 0.736 \\ x(5) &= \frac{10}{e^5} \approx 0.067\end{aligned}$$

$$\begin{aligned}\text{Distance} &= \frac{2}{e} + \frac{2}{e} - \frac{10}{e^5} \\ &\approx 1.404\end{aligned}$$

1984 AB5

Solution

(a) $A = \pi r^2$

$$\text{When } r = 3, \frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 2\pi \cdot 3 \cdot \frac{1}{2} = 3\pi$$

(b) $V = \frac{1}{3}\pi r^2 h$

or

$$V = \frac{1}{3}Ah$$

$$\frac{dV}{dt} = \frac{1}{3}\pi r^2 \frac{dh}{dt} + \frac{2}{3}\pi rh \frac{dr}{dt}$$

$$28\pi = \frac{1}{3}\pi(9) \frac{dh}{dt} + \frac{2}{3}\pi(3)(4)\left(\frac{1}{2}\right)$$

$$\frac{dh}{dt} = 8$$

$$\frac{dV}{dt} = \frac{1}{3}A \frac{dh}{dt} + \frac{1}{3}h \frac{dA}{dt}$$

$$28\pi = \frac{1}{3}(9\pi) \frac{dh}{dt} + \frac{1}{3}4(3\pi)$$

$$\frac{dh}{dt} = 8$$

(c) $\frac{dA}{dh} = \frac{\frac{dA}{dt}}{\frac{dh}{dt}} = \frac{3\pi}{8}$

or

$$A = \pi r^2$$

$$\frac{dA}{dh} = 2\pi r \frac{dr}{dh}$$

$$\frac{dr}{dh} = \frac{\frac{dr}{dt}}{\frac{dh}{dt}} = \frac{\frac{1}{2}}{8} = \frac{1}{16}$$

$$\text{Therefore } \frac{dA}{dh} = 2\pi(3)\left(\frac{1}{16}\right) = \frac{3\pi}{8}$$

1973 AB6**Solution**

$$\text{For } 0 \leq x \leq 3, P(x) = 13x - (x^2 + 5x + 7) = -x^2 + 8x - 7$$

$$\text{For } 3 < x \leq 10, P(x) = 13x - (x^2 + 5x + 7 + 3(x - 3)) = -x^2 + 5x + 2$$

$$\text{Profit function: } P(x) = \begin{cases} -x^2 + 8x - 7, & 0 \leq x \leq 3 \\ -x^2 + 5x + 2, & 3 < x \leq 10 \end{cases}$$

$$P'(x) = \begin{cases} -2x + 8, & 0 < x < 3 \\ -2x + 5, & 3 < x < 10 \end{cases}$$

$P'(x) \neq 0$ for any x in the interval $0 < x < 10$.

However $P'(x) > 0$ for $0 < x < 3$ and so $P(x)$ is increasing on this interval.

Also $P'(x) < 0$ for $3 < x < 10$ and so $P(x)$ is decreasing on this interval.

Therefore P has an absolute maximum at $x = 3$.

This can also be verified by checking the value at $x = 3$ against the values at the endpoints.

$$P(0) = -7$$

$$P(10) = -48$$

$$P(3) = 8$$

1970 AB3/BC2**Solution**

(a) $f(x) = x^{4/3} + 4x^{1/3} = x^{1/3}(x + 4)$

$$f'(x) = \frac{4}{3}x^{1/3} + \frac{4}{3}x^{-2/3} = \frac{4}{3}\left(\frac{x+1}{x^{2/3}}\right)$$

$f'(x) = 0$ at $x = -1$. There is a horizontal tangent at $(-1, -3)$.

(b) There is a vertical tangent at $(0, 0)$.

(c) The absolute maximum and absolute minimum must occur at a critical point or an endpoint. The candidates are

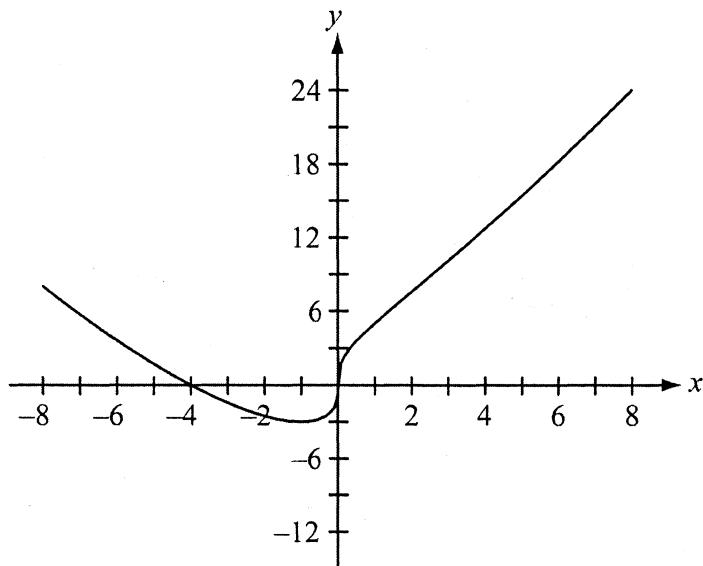
$(-8, 8)$, $(-1, -3)$, $(0, 0)$, and $(8, 24)$

So the absolute maximum is at $(8, 24)$ and the absolute minimum is at $(-1, -3)$.

(d) $f''(x) = \frac{4}{9}x^{-2/3} - \frac{8}{9}x^{-5/3} = \frac{4}{9}\left(\frac{x-2}{x^{5/3}}\right)$

The graph is concave down for $0 < x < 2$.

(e)



1970 AB4
Solution

Method 1:

The combined surface area of the hemisphere and its base is

$$S = \frac{1}{2}(4\pi r^2) + \pi r^2 = 3\pi r^2$$

$$18 = \frac{dS}{dt} = 6\pi r \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{3}{\pi r}$$

Since the height of the cone is $h = r$, the volume of the cone is $V = \frac{1}{3}\pi r^3$

$$\frac{dV}{dt} = \pi r^2 \frac{dr}{dt} = \pi r^2 \left(\frac{3}{\pi r} \right) = 3r$$

$$\text{At } r = 4, \frac{dV}{dt} = 12$$

Method 2:

As in method 1, $S = 3\pi r^2$ and so $V = \frac{1}{3}\pi \left(\frac{S}{3\pi}\right)^{\frac{3}{2}}$.

$$\frac{dV}{dt} = \frac{1}{3}\pi \cdot \frac{3}{2} \left(\frac{S}{3\pi}\right)^{\frac{1}{2}} \cdot \frac{1}{3\pi} \cdot \frac{dS}{dt} = \frac{1}{6} \left(\frac{S}{3\pi}\right)^{\frac{1}{2}} \frac{dS}{dt}$$

$$\text{When } r = 4, S = 48\pi \text{ and so } \frac{dV}{dt} = \frac{1}{6} \cdot 4 \cdot 18 = 12$$

1990 BC3
Solution

(a) $f(x) = 12 - x^2; f'(x) = -2x$

slope of tangent line at

$$(k, f(k)) = -2k$$

line through $(4, 0)$ & $(k, f(k))$ has slope

$$\frac{f(k) - 0}{k - 4} = \frac{12 - k^2}{k - 4}$$

$$\text{so } -2k = \frac{12 - k^2}{k - 4} \Rightarrow k^2 - 8k + 12 = 0$$

$$k = 2 \text{ or } k = 6 \text{ but } f(6) = -24$$

so 6 is not in the domain.

$$k = 2$$

(b) $A = \frac{1}{2}c \cdot f\left(\frac{c}{2}\right) = \frac{1}{2}c\left(12 - \frac{c^2}{4}\right)$

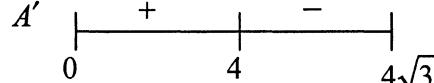
$$= 6c - \frac{c^3}{8} \text{ on } [0, 4\sqrt{3}]$$

$$\frac{dA}{dc} = 6 - \frac{3c^2}{8}; 6 - \frac{3c^2}{8} = 0 \text{ when } c = 4.$$

Candidate test

c	A
0	0
4	16 ← Max
$4\sqrt{3}$	0

First derivative



second derivative

$$\left. \frac{d^2 A}{dc^2} \right|_{c=4} = -3 < 0 \text{ so } c = 4 \text{ gives a relative max.}$$

$c = 4$ is the only critical value in the domain interval, therefore maximum

1975 AB4/BC1

Solution

(a) $y' = 1 + \cos x$

Therefore $x = \pi$ is the only critical point on the interval $-\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$. But $y' \geq 0$ on this interval, hence π is not an extreme point. The minimum and maximum must occur at the endpoints.

$$\text{At } x = -\frac{\pi}{2}, y = -\frac{\pi}{2} + \sin\left(-\frac{\pi}{2}\right) = -\frac{\pi}{2} - 1$$

$$\text{At } x = \frac{3\pi}{2}, y = \frac{3\pi}{2} + \sin\left(\frac{3\pi}{2}\right) = \frac{3\pi}{2} - 1$$

The absolute minimum is at $\left(-\frac{\pi}{2}, -\frac{\pi}{2} - 1\right)$.

The absolute maximum is at $\left(\frac{3\pi}{2}, \frac{3\pi}{2} - 1\right)$.

(b) $y'' = -\sin x$

$y'' = 0$ at $x = 0$ and $x = \pi$.

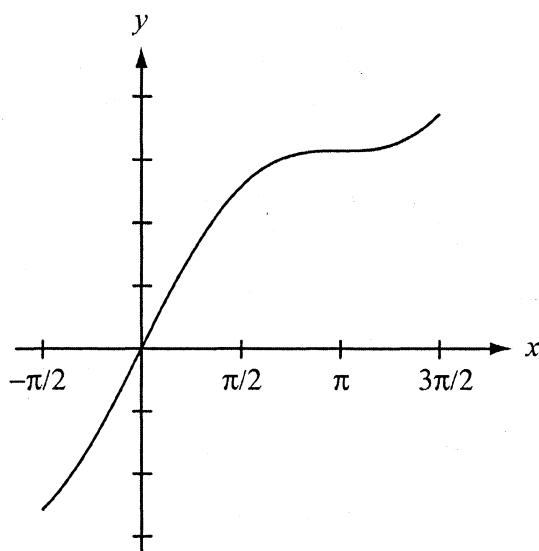
$y'' > 0$ for $-\frac{\pi}{2} < x < 0$

$y'' < 0$ for $0 < x < \pi$

$y'' > 0$ for $\pi < x < \frac{3\pi}{2}$

Therefore $(0, 0)$ and (π, π) are inflection points.

(c)



1978 AB5/BC1**Solution**

(a) Implicit differentiation gives

$$2x - xy' - y + 2yy' = 0$$

$$(2y - x)y' = y - 2x$$

$$y' = \frac{y - 2x}{2y - x}$$

- (b) There is a vertical tangent when $2y - x = 0$, so $x = 2y$. Substituting into the equation of the curve gives $(2y)^2 - (2y)y + y^2 = 9$, or $3y^2 = 9$. Therefore $y = \pm\sqrt{3}$ and the two points on the curve where the tangents are vertical are $(2\sqrt{3}, \sqrt{3})$ and $(-2\sqrt{3}, -\sqrt{3})$.

$$(c) y'' = \frac{(2y - x)(y' - 2) - (y - 2x)(2y' - 1)}{(2y - x)^2}$$

$$\text{At the point } (0, 3), y' = \frac{3-0}{6-0} = \frac{1}{2} \text{ and so } y'' = \frac{(6-0)\left(\frac{1}{2}-2\right)-(3-0)(1-1)}{(6-0)^2} = -\frac{1}{4}$$

Alternatively, one can use implicit differentiation a second time to get

$$2 - xy'' - y' - y' + 2yy' + 2(y')^2 = 0$$

Substituting $x = 0$, $y = 3$, and $y' = \frac{1}{2}$ gives

$$2 - 0 - \frac{1}{2} - \frac{1}{2} + 6y'' + 2\left(\frac{1}{4}\right) = 0 \Rightarrow 6y'' = -\frac{3}{2} \Rightarrow y'' = -\frac{1}{4}$$

1995 AB5/BC3**Solution**

$$(a) \frac{r}{h} = \frac{4}{12} = \frac{1}{3} \quad r = \frac{1}{3}h$$

$$V = \frac{1}{3}\pi \left(\frac{1}{3}h\right)^2 h = \frac{\pi h^3}{27}$$

$$(b) \frac{dV}{dt} = \frac{\pi h^2}{9} \frac{dh}{dt}$$

$$= \frac{\pi h^2}{9} (h - 12) = -9\pi$$

V is decreasing at $9\pi \text{ ft}^3/\text{min}$

(c) Let W = volume of water in cylindrical tank

$$W = 400\pi y; \quad \frac{dW}{dt} = 400\pi \frac{dy}{dt}$$

$$400\pi \frac{dy}{dt} = 9\pi$$

y is increasing at $\frac{9}{400} \text{ ft/min}$