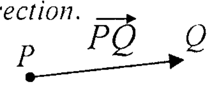
Precalculus Unit 7 Notes—Vectors in the Plane

Quantities that have both magnitude & direction are represented by vectors.

Two vectors are equal if their corresponding directed line segments have the same *length & direction*.
Two vectors are equal if and only if they have the same ***component form***.



DEFINITION --- Component form and Magnitude of a Vector

If v is a vector in the plane equal to the vector with initial point (0, 0) and terminal point (v_1, v_2), then the ***component form*** of v is $$v = \langle v_1, v_2 \rangle$$

Note: *If a vector has initial point* (x_1, y_1) *and terminal point* (x_2, y_2)*, it component form is* $\langle x_2 - x_1, y_2 - y_1 \rangle$.

The ***magnitude*** (or length) of vector $v = \overrightarrow{PQ}$ determined by $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ is

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Note: *If a vector has a length of 0 and no direction, it is the zero vector and is denoted* < 0, 0 >

Ex 1) Find each vector represented by the directed line segment and calculate their magnitudes.

a) Find u, the vector from R = (-4, 2) to S = (-1, 6)

$$\vec{u} = \langle -1-(-4), 6-2 \rangle = \langle 3, 4 \rangle$$

b) Find v, the vector from 0 = (0, 0) to P = (3, 4)

$$\vec{v} = \langle 3, 4 \rangle$$

c) Show that $u = v$. (same component form)

$$|\vec{u}| = \sqrt{(3)^2 + (4)^2} = \sqrt{25} = 5 \qquad |\vec{v}| = \sqrt{(3)^2 + (4)^2} = \sqrt{25} = 5$$

Ex 2) Let u be the vector represented by the directed line segment from $R = (7, -3)$ to $S = (4, -5)$, & v the vector from O (0, 0) to $P = (3, -2)$. Prove that $u = v$.

$$\vec{u} = \langle 4-7, -5-(-3) \rangle = \langle -3, -2 \rangle \qquad |\vec{u}| = \sqrt{(-3)^2 + (-2)^2} = \sqrt{13}$$

$$\vec{v} = \langle -3, -2 \rangle \qquad |\vec{v}| = \sqrt{(-3)^2 + (-2)^2} = \sqrt{13}$$

Ex 3) Find the magnitude of vector R = < 4, -2 >.

$$|R| = \sqrt{(4)^2 + (-2)^2} = \sqrt{20} = 2\sqrt{5}$$

Ex 4) Find the magnitude of vector P = < -3, -5 >.

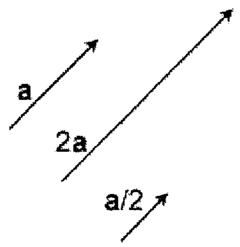
$$|P| = \sqrt{(-3)^2 + (-5)^2} = \sqrt{9 + 25} = \sqrt{34}$$

DEFINITION --- Vector Addition and Scalar Multiplication

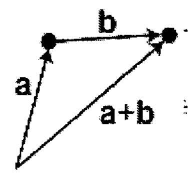
Let $u = \langle u_1, u_2 \rangle$ and $v = \langle v_1, v_2 \rangle$ be vectors and let k be a real number (scalar).

The **sum** (or ***resultant vector***) of $u + v$ is the vector: $u + v = \langle u_1 + v_1, u_2 + v_2 \rangle$

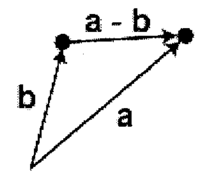
The **scalar product** of vector u and scalar k is the vector: $k\mathbf{u} = k\langle u_1, u_2 \rangle = \langle ku_1, ku_2 \rangle$



scalar product



vector addition



vector subtraction

Ex 5) Let $u = \langle 5, -2 \rangle$ and $v = \langle 6, 4 \rangle$ find the component form each of the following vectors:

a) $u - v$ b) $5u$ c) $3\mathbf{u} + (-2)v$

$\langle 5-6, -2-4 \rangle$ $\quad 5\langle 5, -2 \rangle$ $\quad 3\langle 5, -2 \rangle + -2\langle 6, 4 \rangle$

$\langle -1, -6 \rangle$ $\quad \langle 25, -10 \rangle$ $\quad \langle 15, -6 \rangle + \langle -12, -8 \rangle$

$\langle 3, -14 \rangle$

DEFINITION --- Unit Vector and the standard Unit Vector

A vector u with length 1 is called a unit vector. To create a unit vector u in the direction of v simply divide vector v by its magnitude: $\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|} = \dfrac{1}{|\mathbf{v}|}\mathbf{v}$

The two unit vectors $i = \langle 1, 0 \rangle$ and $j = \langle 0, 1 \rangle$ are the **standard unit vectors** and can be used to write a vector as a linear combination of i & j.

Ex 6) Find a unit vector in the direction of $v = \langle -3, 2 \rangle$, and verify that it has a length equal to 1. Then write the answer in both component form and as a linear combination of the standard unit vectors.

$|v| = \sqrt{(-3)^2 + (2)^2} = \sqrt{13}$ The component form of the vector u a unit vector in the direction of v is $\left\langle \frac{-3}{\sqrt{13}}, \frac{2}{\sqrt{13}} \right\rangle$

u written as a linear combination of the standard unit vectors i & j is $\frac{-3}{\sqrt{13}} i + \frac{2}{\sqrt{13}} j$

Ex 7) Find a unit vector in the direction of $v = \langle 5, -3 \rangle$, and verify that it has a length equal to 1.

$|\vec{v}| = \sqrt{(5)^2 + (-3)^2} = \sqrt{34}$ unit vector: $\left\langle \frac{5}{\sqrt{34}}, \frac{-3}{\sqrt{34}} \right\rangle$

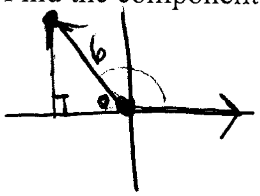
length $= \sqrt{\left(\frac{5}{\sqrt{34}}\right)^2 + \left(\frac{-3}{\sqrt{34}}\right)^2} = \sqrt{\frac{25}{34} + \frac{9}{34}} = \sqrt{\frac{34}{34}} = \sqrt{1} = 1$

DEFINITION --- Direction Angle

To precisely specify the direction of a vector, state its **direction angle** θ (made by the vector and the positive x-axis)

Using trigonometry, we can see the horizontal component of a vector v is ($|v|\cos\theta$) and the vertical component is ($|v|\sin\theta$), thus: $v = (|v|\cos\theta)i + (|v|\sin\theta)j = \langle |v|\cos\theta, |v|\sin\theta \rangle$

Ex 8) Find the components of vector v with direction angle $\theta = 115°$ and magnitude of 6.



$\langle 6\cos 115°, 6\sin 115° \rangle$

$\langle -2.54, 5.44 \rangle$

Ex 9) Find the components of vector v with direction angle $\theta = 230°$ and magnitude of 12.

$\langle 12\cos 230°, 12\sin 230° \rangle$

$\langle -7.71, -9.19 \rangle$

Ex 10) Find the magnitude & direction angle of each vector:

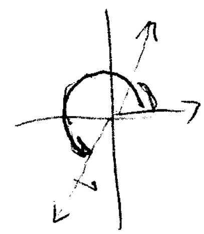
a) $u = \langle 3, 2 \rangle$ quad 1

$|\vec{u}| = \sqrt{(3)^2 + (2)^2} = \sqrt{13}$

$\tan\theta = \frac{2}{3}$

$\theta = \tan^{-1}\left(\frac{2}{3}\right) = 33.69°$

b) $v = \langle -2, -5 \rangle$ quad 3



$|\vec{v}| = \sqrt{(-2)^2 + (-5)^2} = \sqrt{29}$

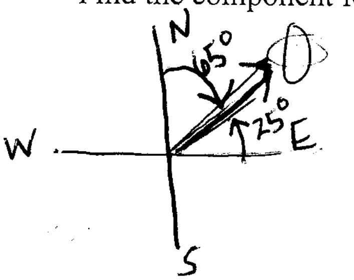
$\tan\theta = \frac{-5}{-2}$

$\theta = \tan^{-1}\left(\frac{5}{2}\right) = 68.20° + 180°$

$\Rightarrow \boxed{248.20°}$

- **The *velocity* of a moving object is a vector because it has both magnitude and direction.**
- **The magnitude of velocity is *speed*.**
- **A *bearing* measures clockwise from the north.**

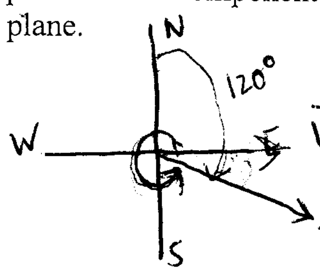
Ex 11) A jet is flying on a bearing of 65° at 500 mph Find the component form of the velocity of the jet.



$\vec{v} = \langle 500\cos 25°, 500\sin 25° \rangle$

$\langle 453.15, 211.31 \rangle$

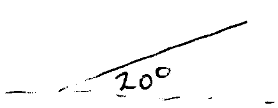
Ex 12) A plane is flying on a bearing of 120° at 425 mph. Find the component form of the velocity of the plane.



$\vec{v} = \langle 425\cos 330°, 425\sin 330° \rangle$

$\langle 368.06, -212.5 \rangle$

Ex 13) Suppose we are pushing a crate up a 20° inclined plane with a force of 3.0 lb. Find the component form of the force.

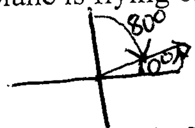


$\langle 3\cos 20°, 3\sin 20° \rangle$

$\langle 2.82, 1.03 \rangle$

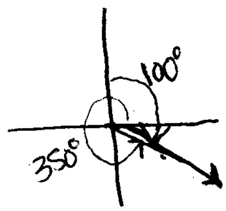
...en 2 forces act upon an object, their **sum** is the ***resultant*** vector showing the velocity of the object itself with ...oth forces acting upon it.

Ex 14) a) An airplane is flying on a bearing of 80° at 540 mph. Find the component form of the velocity of the airplane.



$\vec{p} = \langle 540\cos 10°, 540\sin 10° \rangle = \langle 531.80, 93.77 \rangle$

b) Suppose the plane is flying with a 55 mph wind that has a bearing of 100°. What is the actual ground speed and direction of the plane?

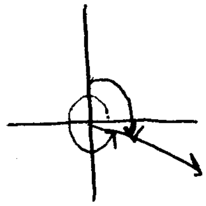


$\vec{w} = \langle 55\cos 350°, 55\sin 350° \rangle = \langle 54.16, -9.55 \rangle$

$\vec{p} + \vec{w} = \langle 585.96, 84.22 \rangle \quad |\vec{p}+\vec{w}| = \sqrt{585.96^2 + 84.22^2} = \boxed{591.98 \text{ mph}}$

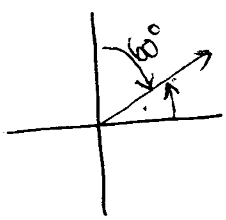
$\tan\theta = \frac{84.22}{585.96} \quad \boxed{\theta = 8.18°}$

Ex 15) A plane is flying on a bearing of 94.14° at 450mph. A wind is blowing on a bearing of 60° at 65 mph. Find the component form of the resultant velocity of the airplane. Then state the actual speed and direction of the plane.



$\vec{p} = \langle 450\cos 355.86, 450\sin 355.86 \rangle$

$\vec{w} = \langle 65\cos 30°, 65\sin 30° \rangle$



$\vec{p} + \vec{w} = \langle 505.12, .01 \rangle$

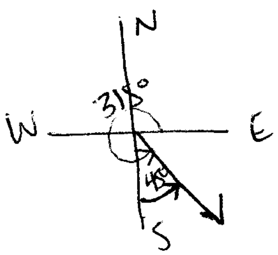
$|\vec{p} + \vec{w}| = \sqrt{505.12^2 + .01^2} = 505.12 \text{ mph}$

$\theta = \tan^{-1}\left(\frac{.01}{505.12}\right) = \boxed{.001°}$

Ex 16) An airplane's velocity with no wind is 580 km/h with a bearing of N60°E. The wind at the altitude of the plane has a velocity of 60 km/h and is blowing SE (which means S45°E). What is the true speed and bearing of the plane?

$\vec{p} = \langle 580\cos 30°, 580\sin 30° \rangle$

$\vec{w} = \langle 60\cos 315°, 60\sin 315° \rangle$



$\vec{p} + \vec{w} = \langle 544.72, 247.57 \rangle$

$|\vec{p} + \vec{w}| = \sqrt{544.72^2 + 247.57^2} = 598.34 \text{ km/h}$

$\theta = \tan^{-1}\left(\frac{247.57}{544.72}\right) = \underbrace{24.44°}_{\text{bearing of } 65.56°}$