Pre-Calculus Name: ______________________

Notes—(6.1) Vectors in the Plane

Quantities that have both magnitude & direction are represented by vectors.

Two vectors are equal if their corresponding directed line segments have the same *length & direction*.
Two vectors are equal if and only if they have the same ***component form***.

$P \longrightarrow Q$, $\overrightarrow{PQ}$

DEFINITION --- Component form and Magnitude of a Vector

If $\boldsymbol{v}$ is a vector in the plane equal to the vector with initial point (0, 0) and terminal point (v_1, v_2), then the ***component form*** of $\boldsymbol{v}$ is $\boldsymbol{v} = \langle v_1, v_2 \rangle$

Note: *If a vector has initial point (x_1, y_1) and terminal point (x_2, y_2), it component form is $\langle x_2 - x_1, y_2 - y_1 \rangle$.*

The ***magnitude*** (or length) of vector $\boldsymbol{v} = \overrightarrow{PQ}$ determined by $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ is

$$|\boldsymbol{v}| = \sqrt{v_1^2 + v_2^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Note: *If a vector has a length of* 0 *and no direction, It is the zero vector and is denoted* $\langle 0,0 \rangle$ $\vec{0}$

Ex 1) Finding the vector represented by the directed line segment and calculate their magnitude.

a) Find $\boldsymbol{u}$ the vector from $R = (-4, 2)$ to $S = (-1, 6)$

$$\vec{u} = \langle -1 - {}^{-}4, 6 - 2 \rangle = \langle 3, 4 \rangle$$

b) Find $\boldsymbol{v}$ the vector from O (0, 0) to $P = (3, 4)$.

$$\vec{v} = \langle 3-0, 4-0 \rangle = \langle 3, 4 \rangle$$

c) Show that $\boldsymbol{u} = \boldsymbol{v}$.

$$|\vec{u}| = \sqrt{(3)^2 + (4)^2} = \sqrt{25} = 5 \qquad |\vec{v}| = \sqrt{(3)^2 + (4)^2} = \sqrt{25} = 5$$

Ex2) Let $\boldsymbol{u}$ be the vector represented by the directed line segment from $R = (7, -3)$ to $S = (4, -5)$, & $\boldsymbol{v}$ the vector from O (0, 0) to $P = (-3, -2)$. Prove $\boldsymbol{u} = \boldsymbol{v}$

$$\vec{u} = \langle 4 - 7, -5 - {}^{-}3 \rangle = \langle -3, -2 \rangle$$

$$\vec{v} = \langle -3 - 0, -2 - 0 \rangle = \langle -3, -2 \rangle$$

$$|\vec{u}| = \sqrt{(-3)^2 + (-2)^2} = \sqrt{13} \qquad |\vec{v}| = \sqrt{(-3)^2 + (-2)^2} = \sqrt{13}$$

Ex3) Find the magnitude of vector R $\langle 4, -2 \rangle$

$$|R| = \sqrt{(4)^2 + (-2)^2} = \sqrt{20} = 2\sqrt{5}$$

Ex4) Find the magnitude of vector P $\langle -3, -5 \rangle$

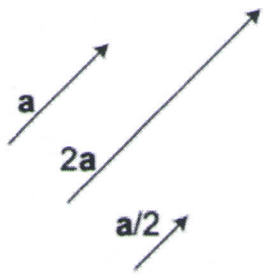
$$|P| = \sqrt{(-3)^2 + (-5)^2} = \sqrt{9 + 25} = \sqrt{34}$$

DEFINITION --- Vector Addition and Scalar Multiplication

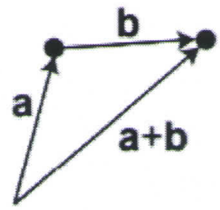
Let $\boldsymbol{u} = \langle u_1, u_2 \rangle$ and $\boldsymbol{v} = \langle v_1, v_2 \rangle$ be vectors and let k be a real number (scalar).

The **sum** (or ***resultant vector***) of $\boldsymbol{u} + \boldsymbol{v}$ is the vector: $\boldsymbol{u} + \boldsymbol{v} = \langle u_1 + v_1, u_2 + v_2 \rangle$

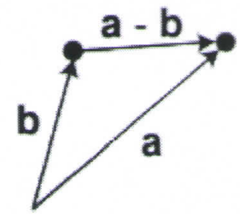
The **scalar product** of vector $\boldsymbol{u}$ and scalar k is the vector: $k\mathbf{u} = k\langle u_1, u_2 \rangle = \langle ku_1, ku_2 \rangle$



Scalar Product



Vector Addition



Vector Subtraction

Ex 5) Let $\boldsymbol{u} = \langle -1, 3 \rangle$ and $\boldsymbol{v} = \langle 4, 7 \rangle$ find the component form each of the following vectors:

a) $\boldsymbol{u} + \boldsymbol{v}$ **b)** $3\boldsymbol{u}$ **c)** $2\mathbf{u} + (-1)\boldsymbol{v}$

a) $\langle -1+4, 3+7 \rangle = \langle 3, 10 \rangle$

b) $\langle 3(-1), 3(3) \rangle = \langle -3, 9 \rangle$

c) $\langle -2, 6 \rangle + \langle -4, -7 \rangle = \langle -6, -1 \rangle$

Ex6) Let $\boldsymbol{u} = \langle 5, -2 \rangle$ and $\boldsymbol{v} = \langle 6, 4 \rangle$ find the component form each of the following vectors:

a) $\boldsymbol{u} - \boldsymbol{v}$ **b)** $5\boldsymbol{u}$ **c)** $3\mathbf{u} + (-2)\boldsymbol{v}$

a) $\langle -5-6, -2-4 \rangle = \langle -1, -6 \rangle$

b) $\langle 5(5), 5(-2) \rangle = \langle 25, -10 \rangle$

c) $\langle 15, -6 \rangle + \langle -12, -8 \rangle = \langle 3, -14 \rangle$

DEFINITION --- Unit Vectors and the standard Unit Vectors

A vector $\boldsymbol{u}$ with length 1 is called a unit vector. to create a unit vector $\boldsymbol{u}$ in the direction of $\boldsymbol{v}$ simply divide vector $\boldsymbol{v}$ by its magnitude: $\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|} = \dfrac{1}{|\mathbf{v}|}\mathbf{v}$

The two unit vectors $\boldsymbol{i} = \langle 1, 0 \rangle$ and $\boldsymbol{j} = \langle 0, 1 \rangle$ are the **standard unit vectors** and can be used to write a vector as a linear combination of $\boldsymbol{i}$ & $\boldsymbol{j}$.

Ex7) Find a unit vector in the direction of $\boldsymbol{v} = \langle -3, 2 \rangle$, and verify that it has a length equal to 1. Then write the answer in both component form and as a linear combination of the standard unit vectors.

$|\boldsymbol{v}| = \sqrt{(-3)^2+(2)^2} = \sqrt{13}$ The component form of the vector $\boldsymbol{u}$ a unit vector in the direction of $\boldsymbol{v}$ is $\left\langle \dfrac{-3}{\sqrt{13}}, \dfrac{2}{\sqrt{13}} \right\rangle$

$\boldsymbol{u}$ written as a linear combination of the standard unit vectors $\boldsymbol{i}$ & $\boldsymbol{j}$ is $\dfrac{-3}{\sqrt{13}}i + \dfrac{2}{\sqrt{13}}j$

Ex 8) Find the unit vector in the direction of $\boldsymbol{v} = \langle 5, -3 \rangle$, and verify that it has a length equal to 1.

$|\vec{v}| = \sqrt{(5)^2+(-3)^2} = \sqrt{34}$

unit vector: $\left\langle \dfrac{5}{\sqrt{34}}, \dfrac{-3}{\sqrt{34}} \right\rangle$

length $= \sqrt{\left(\dfrac{5}{\sqrt{34}}\right)^2 + \left(\dfrac{-3}{\sqrt{34}}\right)^2} = \sqrt{\dfrac{25}{34} + \dfrac{9}{34}}$

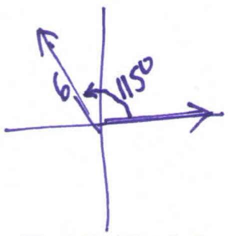
$= \sqrt{\dfrac{34}{34}} = \sqrt{1} = 1$

DEFINITION --- Direction Angle

To precisely specify the direction of a vector state its **direction angle** θ (made by the vector and the positive x-axis)

Using trigonometry, we can see the horizontal component of a vector $\boldsymbol{v}$ is ($|\boldsymbol{v}|\cos\theta$) and the vertical component is ($|\boldsymbol{v}|\sin\theta$), thus: $\boldsymbol{v} = (|\boldsymbol{v}|\cos\theta)\boldsymbol{i} + (|\boldsymbol{v}|\sin\theta)\boldsymbol{j} = \langle |\boldsymbol{v}|\cos\theta, |\boldsymbol{v}|\sin\theta\rangle$

Ex9) Find the components of vector $\boldsymbol{v}$ with direction angle $\theta = 115°$ and magnitude of 6



$\langle 6\cos 115°, 6\sin 115°\rangle$

$\langle -2.54, 5.44\rangle$

Ex10) Find the components of vector v with direction angle $\theta = 230°$ and magnitude of 12.

$\langle 12\cos 230°, 12\sin 230°\rangle$

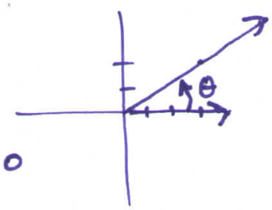
$\langle -7.71, -9.19\rangle$

Ex11) Find the magnitude & direction angle of each vector:

a) $\boldsymbol{u} = \langle 3, 2\rangle$

$|\vec{u}| = \sqrt{3^2 + 2^2} = \sqrt{13}$

$\theta = \tan^{-1}\left(\frac{2}{3}\right) = 33.69°$



b) $\boldsymbol{v} = \langle -2, -5\rangle$

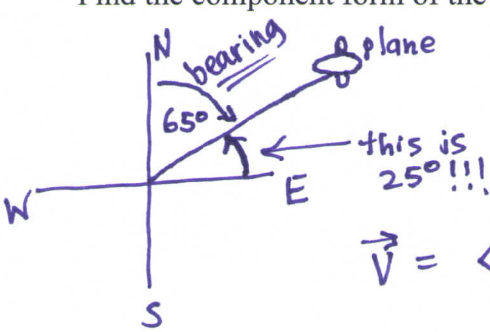
$|\vec{v}| = \sqrt{(-2)^2 + (-5)^2} = \sqrt{29}$

$\theta = \tan^{-1}\left(\frac{-5}{-2}\right) = 68.20°$ 1st quad. $+180°$ need 3rd quad. $\boxed{248.20°}$

The ***velocity*** of a moving object is a vector because it has both magnitude and direction. The magnitude of velocity is ***speed***.

* a "bearing" measures from North and moves clockwise

Ex12) A jet is flying on a bearing of 65° at 500 mph Find the component form of the velocity of the plane.

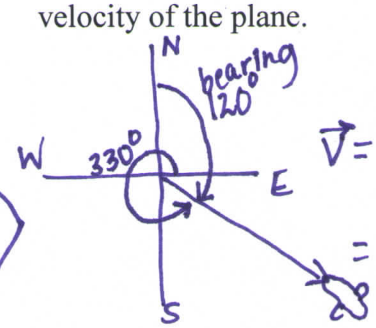


$\vec{V} = \langle 500\cos 25°, 500\sin 25°\rangle$

$\langle 453.12, 211.31\rangle$

Ex13) A plane is flying on a bearing of 120° at 425 mph. Find the component form of the velocity of the plane.

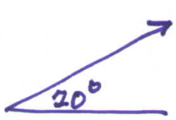
*



$\vec{V} = \langle 425\cos 330°, 425\sin 330°\rangle$

$= \langle 368.06, -212.5\rangle$

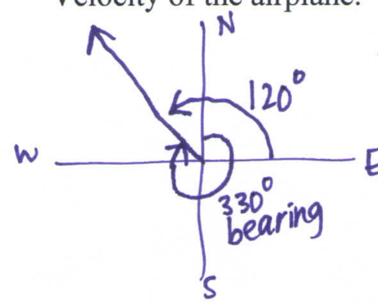
Ex14) Suppose we are pushing a crate up a 20° inclined plane with a force of 3.0 lb. Find the component form of the force.



$\langle 3\cos 20°, 3\sin 20°\rangle$

$\langle 2.82, 1.03\rangle$

Ex 15) An airplane is flying on a bearing of 330° at 500 mph. Find the component form of the Velocity of the airplane.

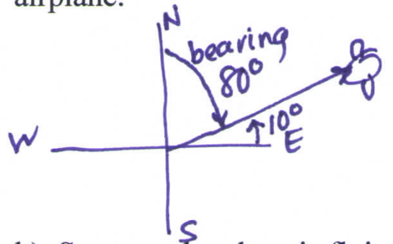


$\vec{V} = \langle 500\cos 120°, 500\sin 120°\rangle$

$= \langle -250, 433.01\rangle$

When 2 forces act upon an object their sum is the *resultant* vector showing the velocity of the object itself with both forces acting upon it.

Ex16) a) An airplane is flying on a bearing of 80° at 540mph. Find the component form of the velocity of the airplane.



$\vec{p} = \langle 540\cos 10°, 540\sin 10° \rangle$

$= \langle 531.80, 93.77 \rangle$

$\vec{p} + \vec{w} = \langle 585.96, 84.22 \rangle$

speed $= \sqrt{(585.96)^2 + (84.22)^2}$

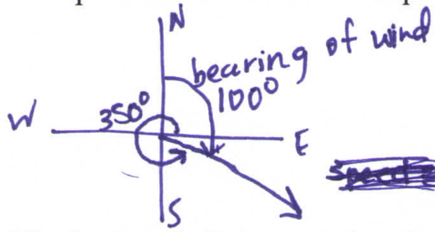
$= 591.98$ mph

$\theta = \tan^{-1}\left(\frac{84.22}{585.96}\right) = 8.18°$

bearing of 81.82°

b) Suppose the plane is flying with a 55mph wind that has a bearing of 100°. What is the actual ground speed and direction of the plane?

j



$\vec{w} = \langle 55\cos 350°, 55\sin 350° \rangle = \langle 54.16, -.87 \rangle$

~~speed = $\sqrt{(54.16)^2 + (.87)^2}$~~

note: $\langle 55\cos(-10°), 55\sin(-10°) \rangle$ gives the same vector

Ex17) A plane is flying on a bearing of 94.14˚ at 450mph. A wind is blowing on a bearing of 60˚ at65 mph. Find the component form of the resultant velocity of the airplane. Then state the actual speed and direction of the plane.

plane: $\langle 450\cos(-4.14°), 450\sin(-4.14°) \rangle$

wind: $\langle 65\cos(30°), 65\sin(30°) \rangle$

$\vec{p} + \vec{w} = \langle 505.12, .01 \rangle$

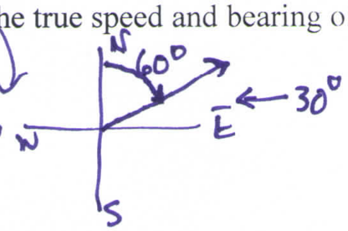
direction $= .001°$ (almost due east)

speed $= \sqrt{(505.12)^2 + (.01)^2}$

$= 501.12$ mph

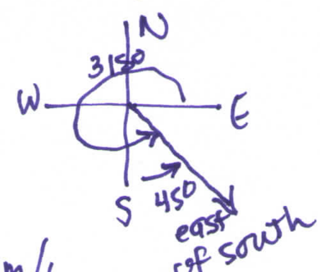
Ex18) An airplanes velocity with no wind is 580 km/h with a bearing of N60˚E. The wind at the altitude of the plane has velocity of 60km/h and is blowing SE (which means S45˚E). What is the true speed and bearing of the plane?

read as "60° east of north"



$\vec{p} = \langle 580\cos 30°, 580\sin 30° \rangle$

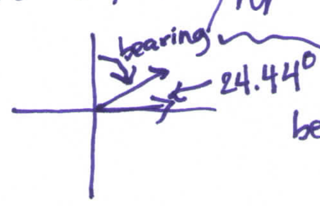
$\vec{w} = \langle 60\cos 315°, 60\sin 315° \rangle$



$\vec{p} + \vec{w} = \langle 544.72, 247.57 \rangle$

speed $= \sqrt{(544.72)^2 + (247.57)^2} = 598.34$ km/hr

$\tan\theta = \frac{247.57}{544.72}$ $\quad \theta = 24.44°$



bearing = N65.56° E

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Notes—(6.2) Dot Product of Vectors

Objectives: You will be able to calculate dot products, the angle between two vectors, and projections of vectors.

DEFINITION --- Dot product

The ***dot product*** or ***inner product*** of $\mathbf{u} = \langle u_1, u_2\rangle$ and $\mathbf{v} = \langle v_1, v_2\rangle$ is → $\mathbf{u}\bullet\mathbf{v} = u_1v_1 + u_2v_2$

Ex1) Find each dot product:

a) $\langle 3, 4\rangle \bullet \langle 5, 2\rangle$

$3\cdot5 + 4\cdot2$

$15 + 8$

23

b) $\langle 1, -2\rangle \bullet \langle -4, 3\rangle$

$1\cdot-4 + -2\cdot3$

$-4 + -6$

-10

c) $(2\mathbf{i} - \mathbf{j}) \bullet (3\mathbf{i} - 5\mathbf{j})$

means $\langle 2,-1\rangle \bullet \langle 3,-5\rangle$

$2\cdot3 + -1\cdot-5$

$6 + 5$

11

Properties of the Dot product --------- Let **u**, **v**, and **w** be vectors and let c be a scalar.

1. $\mathbf{u}\bullet\mathbf{v} = \mathbf{v}\bullet\mathbf{u}$
2. $\mathbf{u}\bullet\mathbf{u} = |\mathbf{u}|^2$
3. $\mathbf{0}\bullet\mathbf{u} = 0$
4. $\mathbf{u}\bullet(\mathbf{v}+\mathbf{w}) = \mathbf{u}\bullet\mathbf{v} + \mathbf{u}\bullet\mathbf{w}$
   $(\mathbf{u}+\mathbf{v})\bullet\mathbf{w} = \mathbf{u}\bullet\mathbf{w} + \mathbf{v}\bullet\mathbf{w}$
5. $(c\mathbf{u})\bullet\mathbf{v} = \mathbf{u}\bullet(c\mathbf{v}) = c(\mathbf{u}\bullet\mathbf{v})$

Ex2) Use the dot product to find the length of vector $\mathbf{u} = \langle 4, -3\rangle$

Since $\vec{u}\cdot\vec{u} = |\vec{u}|^2$ then $\sqrt{\vec{u}\cdot\vec{u}} = |\vec{u}|$

$|\vec{u}| = \sqrt{\langle 4,-3\rangle\cdot\langle 4,-3\rangle} = \sqrt{4\cdot4 + -3\cdot-3} = \sqrt{16+9} = \sqrt{25} = 5$

THEOREM --------- Angle Between Two Vectors $\cos\theta = \dfrac{u\cdot v}{|u||v|}$ and $\theta = \cos^{-1}\left(\dfrac{u\cdot v}{|u||v|}\right)$

Ex3) Find the angle between two vectors **u** & **v**.

a) $\mathbf{u} = \langle 2, 3\rangle$, $\mathbf{v} = \langle -2, 5\rangle$

$$\cos\theta = \frac{\langle 2,3\rangle\cdot\langle -2,5\rangle}{\sqrt{(2)^2+(3)^2}\cdot\sqrt{(-2)^2+(5)^2}}$$

$$\cos\theta = \frac{2\cdot-2 + 3\cdot5}{\sqrt{13}\cdot\sqrt{29}}$$

$\theta = \cos^{-1}\left(\frac{11}{\sqrt{377}}\right)$ $\boxed{\theta = 55.49^\circ}$

b) $\mathbf{u} = \langle 2, 1\rangle$, $\mathbf{v} = \langle -1, -3\rangle$

$$\cos\theta = \frac{2\cdot-1 + 1\cdot-3}{\sqrt{2^2+1^2}\cdot\sqrt{(-1)^2+(-3)^2}} = \frac{-5}{\sqrt{5}\cdot\sqrt{10}}$$

$\theta = \cos^{-1}\left(\frac{-5}{\sqrt{50}}\right)$

$\theta = 135^\circ$

Definition --------- Orthogonal Vectors → The vectors **u** and **v** are orthogonal if and only if **u** • **v** = 0

Ex4) Prove that the vectors **u** = ⟨2, 3⟩ and **v** = ⟨−6, 4⟩ are orthogonal.

$$2\cdot -6 + 3\cdot 4$$
$$-12+12$$
$$0$$

Projection of u onto v ------- If **u** and **v** are nonzero vectors, the projection of **u** onto **v** is $\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}$

Ex5) Find the vector projection of **u** = ⟨6, 2⟩ onto **v** = ⟨5, −5⟩. Then write **u** as the sum of two orthogonal vectors, one of which is proj$_v$**u**.

$$\text{proj}_v u = \frac{6\cdot 5 + 2\cdot -5}{\left(\sqrt{(5)^2+(-5)^2}\right)^2}\cdot\langle 5,-5\rangle$$
$$\frac{20}{50}\cdot\langle 5,-5\rangle$$
$$\langle 2,-2\rangle$$

$$\vec{u} = \text{proj}_v u + \vec{w}$$
$$\vec{u} = \langle 6,2\rangle = \langle 2,-2\rangle + \langle 4,4\rangle$$

dot product = 0

$$2\cdot 4 + -2\cdot 4$$
$$0$$

Ex6) Find the vector projection of **u** = ⟨−3, 4⟩ onto **v** = ⟨12, −5⟩. Then write **u** as the sum of two orthogonal vectors, one of which is proj$_v$**u**.

$$\text{proj}_v u = \frac{-3\cdot 12 + 4\cdot -5}{\left(\sqrt{12^2+(-5)^2}\right)^2}\cdot\langle 12,-5\rangle$$
$$= \left\langle -\frac{672}{169}, \frac{280}{169}\right\rangle$$

$$\vec{u} = \text{proj}_v u + \vec{w}$$
$$\vec{u} = \langle -3,4\rangle = \left\langle \frac{-672}{169}, \frac{280}{169}\right\rangle + \left\langle \frac{165}{169}, \frac{396}{169}\right\rangle$$

$$x + \frac{-672}{169} = -3$$
$$y + \frac{280}{169} = 4$$

~~**Ex7)** Given **v** = **i** - 2**j** + 2**k** and **u** = 4**i** - 3**k** find the vector projection of **u** onto **v**. Then write **u** as the sum of two orthogonal vectors, one of which is proj$_v$**u**.~~