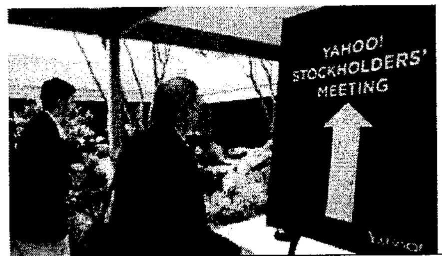
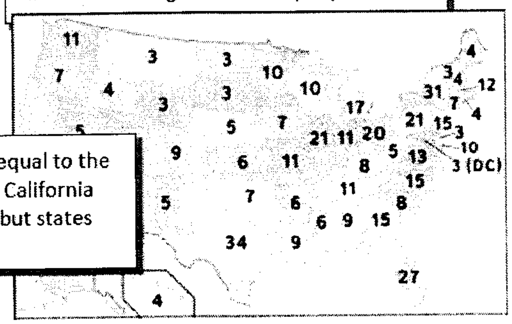
Notes--Weighted Voting

ONE PERSON - ONE VOTE is a democratic idea of equality.
But what if the voters are not PEOPLE but are governments? countries? states?
If the institutions are not equal, then the number of votes they control should not be equal.

The United Nations Security Council – 15 voting nations: 5 permanent members (Britain, China, France, Russia, United States), 10 nonpermanent members appointed for a 2-year rotation. Permanent members have more "votes" than non permanent members.



Stock Holders/Shareholders: The more stock you own, the more say you have in decision making for the company.



The Electoral College—Each state gets a number of "electors" (votes) equal to the number of Senators plus the number of Representatives in Congress. California has 55 votes but North Dakota only has 3 votes. Each state is a voter but states with heavy concentration of population receive a bigger "vote".

weighted voting -- the situation where each voter is not equal in the number of votes they control

- we are interested in how much weight a voter has
- sometimes used to vote on candidates, but more commonly to decide "yes" or "no" on a proposal

Important Terms for Weighted Voting

motion -- a vote with only two choices. (usually yes/no)

voter/player -- each individual or entity casting a vote (symbolized by P_1, P_2, P_3, etc)

weight -- the number of votes each player controls

quota -- the smallest number of votes required to pass a "motion"

A weighted voting system will often be represented in a shorthand form: $[q: w_1, w_2, w_3, \ldots, w_n]$
where q = quota and the w's = weight of each voter.

Example 1 Determine the number of voters, the total # of votes, the minimum # of votes required to pass a resolution, and the number of votes controlled by each player.

a. [14: 8, 6, 5, 1]

# of voters = 4 total # of votes = 20 quota = 14

Player 1 (P_1) = controls 8 votes which means P_1 "has a weight of 8"

Player 2 (P_2) = controls 6 votes

Player 3 (P_3) = controls 5 votes

Player 4 (P_4) = controls 1 votes

b. [51: 26, 26, 12, 12, 12, 12]

# of voters = 6 total # of votes = 100 quota = 51

Player 1 (P_1) = controls 26 votes which means P_1 "has a weight of 26"

Player 2 (P_2) = controls 26 votes

Player 3 (P_3) = controls 12 votes

Player 4 (P_4) = controls 12 votes

Player 5 (P_5) = controls 12 votes

Player 6 (P_6) = controls 12 votes

Common Types of Quotas . . .

Simple majority/Strict majority → $q > \frac{1}{2}$ of total # votes

Two-thirds majority → $q \geq \frac{2}{3}$ of total # votes

Unanimity → $q =$ total # votes

> **U.S. Senate:**
> Simple Majority to pass an ordinary law (51 votes)
>
> 60 votes to stop a filibuster
>
> 2/3 of the votes (67) to override a presidential veto.

For a weighted voting system to be legal: the quota must be at least a Simple majority and no more than total # of votes.

Symbolically: If $V = w_1 + w_2 + w_3 + \ldots + w_n$, then $\frac{V}{2} < q \leq V$

Example 2 Suppose you are given the voting system [q: 7, 2, 1, 1, 1].

5 voters

12 votes

$\frac{1}{2}(12) = 6$

a. What is the smallest legal quota? 7

b. What is the largest legal quota? 12

14

c. What is the value of the quota if *at least* two-thirds of the votes are required to pass a motion? $\frac{2}{3}(12) = 8$

d. What is the value of the quota if more than *three-fourths* of the votes are required to pass? $\frac{3}{4}(12) = 9$ $q > 9$

Example 3 Four partners decide to start a business. P_1 buys 8 shares, P_2 buys 7 shares, P_3 buys 3 shares and P_4 buys 2 shares. Suppose that one share = one vote. [q: 8, 7, 3, 2]

total votes = 20

a. The quota is set at two-thirds of the total number of votes. Describe as a weighted voting system.

$\frac{2}{3}(20) = 13\frac{1}{3}$ [14: 8, 7, 3, 2]

b. The partnership above decides the quota is too high and changes the quota to 10 votes. Describe as a weighted voting system and determine the problem associated with this situation. [10: 8, 7, 3, 2] illegal system – quota is less than a simple majority

c. The partnership above decides to make the quota equal to 21 votes. Describe as a weighted voting system and determine the problem associated with this situation. [21: 8, 7, 3, 2] illegal system – quota is more than the # of votes

d. What if our partnership changed the quota to 19? Describe as a weighted voting system and determine what happens with this situation.

[19: 8, 7, 3, 2] all have to vote same way to pass a resolution

A Look at Power . . .

Example 4 Suppose you are given the voting system [11: 12, 5, 4]. What do you notice about P_1?

P_1 has all the power ⇒ dictator

P_2 & P_3 have no power ⇒ dummies

> Note:
> If any player is a dictator, then EVERY OTHER PLAYER is a dummy.
>
> Even if there is no dictator, there may still be dummies.

Example 5 Suppose you are given the voting system [30: 10, 10, 10, 9]. What are the winning groups? Are there any dummies (players not needed)?

$P_1 + P_2 + P_3 = 30$

$P_1 + P_2 + P_3 + P_4 = 39$

P_4 is a dummy

Example 6 Suppose you are given the voting system [12: 9, 5, 4, 2]. Is there a dictator? no If P_1 chooses to vote against the motion, can the other players combine weight to meet the quota? no

P_1 has "veto power"

NOTE: If a player is not a dictator, but the other players cannot meet the quota without his votes, we say he has veto power. Sometimes, more than one player will have veto power.

Summary

dictator -- if their weight is equal to or greater than the quota; can block any proposal from passing because the other players cannot reach quota without the dictator

veto power -- the quota can only be reached if a certain player is in support of the proposal

dummy -- a player who has no influence in the outcome

Example 7 Identify if any players are dictators or dummies and if any player has veto power.

a. [15: 16, 8, 4, 1]

P_1 dictator

P_2, P_3, P_4 dummies

b. [18: 16, 8, 4, 1]

no dictator

P_4 dummy

P_1 has veto power

Who is the most POWERFUL player?

Coalition -- a group of voters who choose to vote the same way

weight of the coalition -- add together the weights of the voters in the coalition

winning coalition -- a coalition whose combined weight is greater than or equal to the quota

losing coalition -- a coalition whose combined weight is less than the quota

critical voter -- any player who must be present in a winning coalition for it to remain a winning coalition (in other words . . . if he/she were to leave the coalition, then the coalition would no longer be a "winning" coalition)

Given the voting system [16: 7, 6, 3, 3, 2],

- $\{P_1, P_2, P_4\}$ would represent the coalition of players 1, 2 and 4
- combined weight of 7 + 6 + 3 = 16, which meets quota, so this would be a winning coalition
- every player is critical because leaving the coalition would change it from a winning coalition to a losing coalition

Example 8 [16: 7, 6, 3, 3, 2] Who is critical within the coalitions specified below?

a. $\{P_3, P_4, P_5\}$
3 3 2
8 is wt. of the coalition
losing coalition

b. $\{P_1, P_2, P_3, P_4, P_5\}$
7 6 3 3 2
wt = 21 winning coalition
critical voters: P_1, P_2

Example 9 In the voting system [14: 18, 10, 5] (P_1 P_2 P_3), list all of the possible coalitions. Then determine if any voter is critical to each coalition.

Coalition	weight
P_1	18
P_2	10
P_3	5
P_1, P_2	28

C	W
P_1, P_3	23
P_2, P_3	15
P_1, P_2, P_3	33

winning coal.
P_1
P_1 P_2
P_1 P_3
P_2, P_3
P_1, P_2, P_3

Calculating Power: Banzhaf Power Index

The **Banzhaf power index** was originally created in 1946 by Lionel Penrose, but was reintroduced by John Banzhaf in 1965. The power index is a numerical way of looking at power in a weighted voting situation. A player's power is proportional to the number of **coalitions** for which that player is **critical**. The more often a player is critical, the more power he holds.

Banzhaf power index is calculated by:
1) List all winning coalitions
2) In each coalition, identify the players who are critical
3) Count up how many times each player is critical
4) Convert these counts to fractions or decimals by dividing by the total times any player is critical

Note: The Banzhaf Power DISTRIBUTION for the weighted voting system is the % of power each player holds.

Example 10 Consider the system [16: 7, 6, 3, 3, 2]. The winning coalitions are listed below.

$\{P_1, P_2, P_3\}$
$\{P_1, P_2, P_4\}$
$\{P_1, P_2, P_3, P_4\}$
$\{P_1, P_2, P_3, P_5\}$
$\{P_1, P_2, P_4, P_5\}$
$\{P_1, P_2, P_3, P_4, P_5\}$

Calculate the Banzhaf power index and the Banzhaf power distribution of each voter.