Example 8 [16: 7, 6, 3, 3, 2] Who is critical within the coalitions specified below?

a. {P_3, P_4, P_5}
3 3 2
8 is wt. of the coalition
losing coalition

b. {P_1, P_2, P_3, P_4, P_5}
7 6 3 3 2
wt = 21 winning coalition
critical voters: P_1, P_2

Example 9 In the voting system [14: 18, 10, 5], list all of the possible coalitions. Then determine if any voter is critical to each coalition.

P_1 P_2 P_3

Coalition	weight
P_1	18
P_2	10
P_3	5
P_1, P_2	28

C	W
P_1, P_3	23
P_2, P_3	15
P_1, P_2, P_3	33

winning coal.
P_1
P_1 P_2
P_1 P_3
P_2 P_3
P_1, P_2, P_3

Calculating Power: Banzhaf Power Index

The **Banzhaf power index** was originally created in 1946 by Lionel Penrose, but was reintroduced by John Banzhaf in 1965. The power index is a numerical way of looking at power in a weighted voting situation. A player's power is proportional to the number of **coalitions** for which that player is **critical**. The more often a player is critical, the more power he holds.

Banzhaf power index is calculated by:
1) List all winning coalitions
2) In each coalition, identify the players who are critical
3) Count up how many times each player is critical
4) Convert these counts to fractions or decimals by dividing by the total times any player is critical

Note: The Banzhaf Power DISTRIBUTION for the weighted voting system is the % of power each player holds.

Example 10 Consider the system [16: 7, 6, 3, 3, 2]. The winning coalitions are listed below.

	weight	critical voter
{P_1, P_2, P_3}	16	P_1, P_2, P_3
{P_1, P_2, P_4}	16	P_1, P_2, P_4
{P_1, P_2, P_3, P_4}	19	P_1, P_2
{P_1, P_2, P_3, P_5}	18	P_1, P_2, P_3
{P_1, P_2, P_4, P_5}	18	P_1, P_2, P_4
{P_1, P_2, P_3, P_4, P_5}	21	P_1, P_2

Calculate the Banzhaf power index and the Banzhaf power distribution of each voter.

P_1: $\frac{6}{16} = \frac{3}{8}$ 37.5%

P_2: $\frac{6}{16} = \frac{3}{8}$ 37.5%

P_3: $\frac{2}{16} = \frac{1}{8}$ 12.5%

P_4: $\frac{2}{16} = \frac{1}{8}$ 12.5%

P_5: $\frac{0}{16} = 0$ 0%

Example 11 Consider the system [5: 3, 2, 2]. Calculate the Banzhaf power index of each voter.

Winning Coalitions	weight	critical voter
P_1, P_2	5	P_1, P_2
P_1, P_3	5	P_1, P_3
P_1, P_2, P_3	7	P_1

$P_1: \frac{3}{5}$

$P_2: \frac{1}{5}$

$P_3: \frac{1}{5}$

Banzhaf Coalitions: 3 Players

$\{P_1\}$ 3	$\{P_1,P_2\}$ 5	$\{P_1, P_2, P_3\}$ 7
$\{P_2\}$ 2	$\{P_1,P_3\}$ 5	
$\{P_3\}$ 2	$\{P_2,P_3\}$ 4	

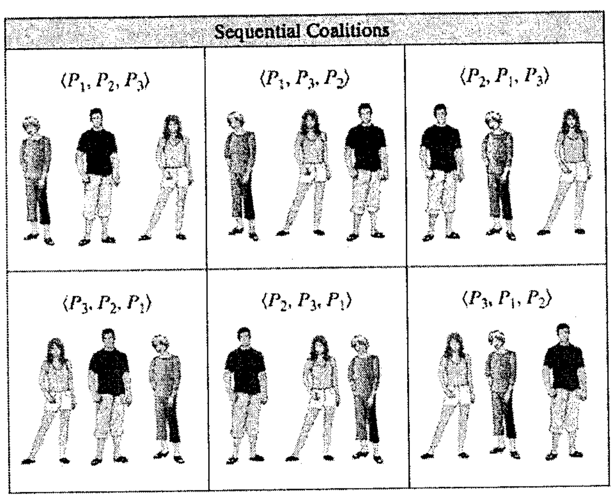
Helpful Hint:

> If n = number of players in a weighted voting system,
>
> Then the number of possible coalitions is: $2^n - 1$

Calculating Power: Shapley-Shubik Power Index

The **Shapley-Shubik power index** was formulated by Lloyd Shapley and Martin Shubik in 1954 to measure the powers of players in a voting game. The Shapley-Shubik power index states that a player's power is proportional to the number of sequential-coalitions for which that player is pivotal. The more times a player is pivotal, the more power he holds.

sequential coalition -- a group of voters in which the order of voters matters.



Factorials:

If N= the number of players, then the number of sequential coalitions is N!

N! = N x (N-1) x . . . x 3 x 2 x 1

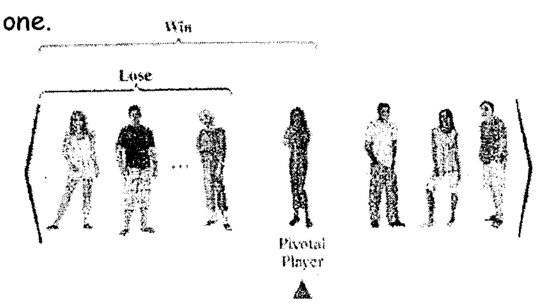
Banzhaf: $\{P_1, P_2, P_3\}$

These 3 players decide to vote together.
They form a coalition.
Order listed in the { } doesn't matter.

Shapley-Shubik: $\langle P_1, P_3, P_2\rangle$

These 3 players decide to vote together.
P_1 votes 1st, P_3 votes 2nd, P_2 votes 3rd.
They form a sequential coalition.
Order listed in the ⟨ ⟩ is important.

pivotal player -- the player in a sequential coalition whose immediate sequential presence changes a losing vote to a winning one.



Example 12 Given the weighted voting system [5: 3,2,1,1}, find the pivotal player for each given sequential coalition.

a. $[P_1,P_4,P_3,P_2]$
3 1 1 2

P_3

b. $[P_3,P_1,P_2,P_4]$
1 3 2 1

P_2

c. $[P_4,P_3,P_2,P_1]$
1 1 2 3

P_1

Example 13 a. List the possible sequences for 3 players. How many are there?

P_1, P_2, P_3 P_2, P_1, P_3 P_3, P_1, P_2 $3! = 3 \cdot 2 \cdot 1 = 6$

P_1, P_3, P_2 P_2, P_3, P_1 P_3, P_2, P_1

b. How many possible sequences for 4 players? for 5 players?

$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

To find a Shapley-Shubik Power Index:

Step 1: Make a list of all sequential coalitions

Step 2: For each sequential coalition, determine the pivotal player.

Step 3: For each player, count the number of times they are pivotal and divide by the number of sequential coalitions. NOTE: Calculate the % if you are asked for the *distribution.*

Example 14 Consider the system [5: 3, 2, 2]. Calculate the Shapley-Shubik power index and the Shapley-Shubik power distribution of each voter.

$P_1: \frac{4}{6}$ $66.\overline{6}\%$

$P_2: \frac{1}{6}$ $16.\overline{6}\%$

$P_3: \frac{1}{6}$ $16.\overline{6}\%$

Sequential Coalitions: 3 Players

$[P_1,P_2,P_3]$
$[P_1,P_3,P_2]$
$[P_2,P_1,P_3]$
$[P_2,P_3,P_1]$
$[P_3,P_1,P_2]$
$[P_3,P_2,P_1]$

Example 15 Find the Shapley-Shubik power distribution for [6: 4, 3, 2, 1].

$P_1: \frac{10}{24}$ $41.\overline{6}\%$

$P_2: \frac{6}{24}$ 25%

$P_3: \frac{6}{24}$ 25%

$P_4: \frac{2}{24}$ $8.\overline{3}\%$

Sequential Coalitions: 4 Players

$[P_1,P_2,P_3,P_4]$	$[P_2,P_1,P_3,P_4]$	$[P_3,P_1,P_2,P_4]$	$[P_4,P_1,P_2,P_3]$
$[P_1,P_2,P_4,P_3]$	$[P_2,P_1,P_4,P_3]$	$[P_3,P_1,P_4,P_2]$	$[P_4,P_1,P_3,P_2]$
$[P_1,P_3,P_2,P_4]$	$[P_2,P_3,P_1,P_4]$	$[P_3,P_2,P_1,P_4]$	$[P_4,P_2,P_1,P_3]$
$[P_1,P_3,P_4,P_2]$	$[P_2,P_3,P_4,P_1]$	$[P_3,P_2,P_4,P_1]$	$[P_4,P_2,P_3,P_1]$
$[P_1,P_4,P_2,P_3]$	$[P_2,P_4,P_1,P_3]$	$[P_3,P_4,P_1,P_2]$	$[P_4,P_3,P_1,P_2]$
$[P_1,P_4,P_3,P_2]$	$[P_2,P_4,P_3,P_1]$	$[P_3,P_4,P_2,P_1]$	$[P_4,P_3,P_2,P_1]$